Shaping up! Introduction into Shape Analysis

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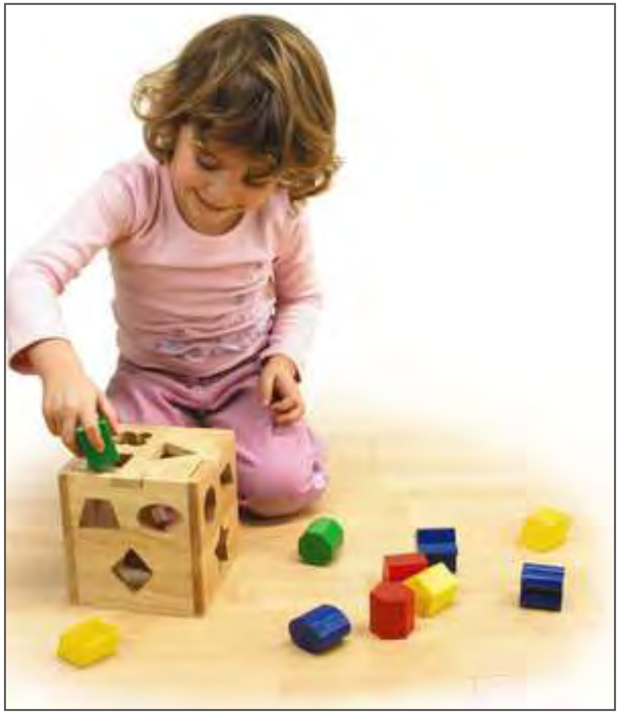
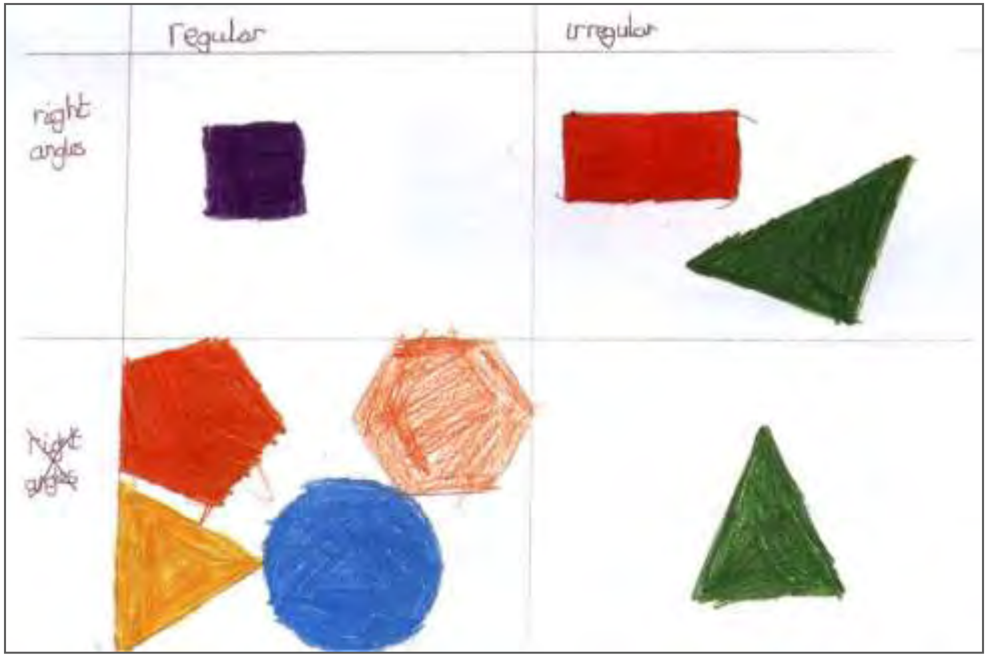
Monreale Cathedral: Pavement

Shape

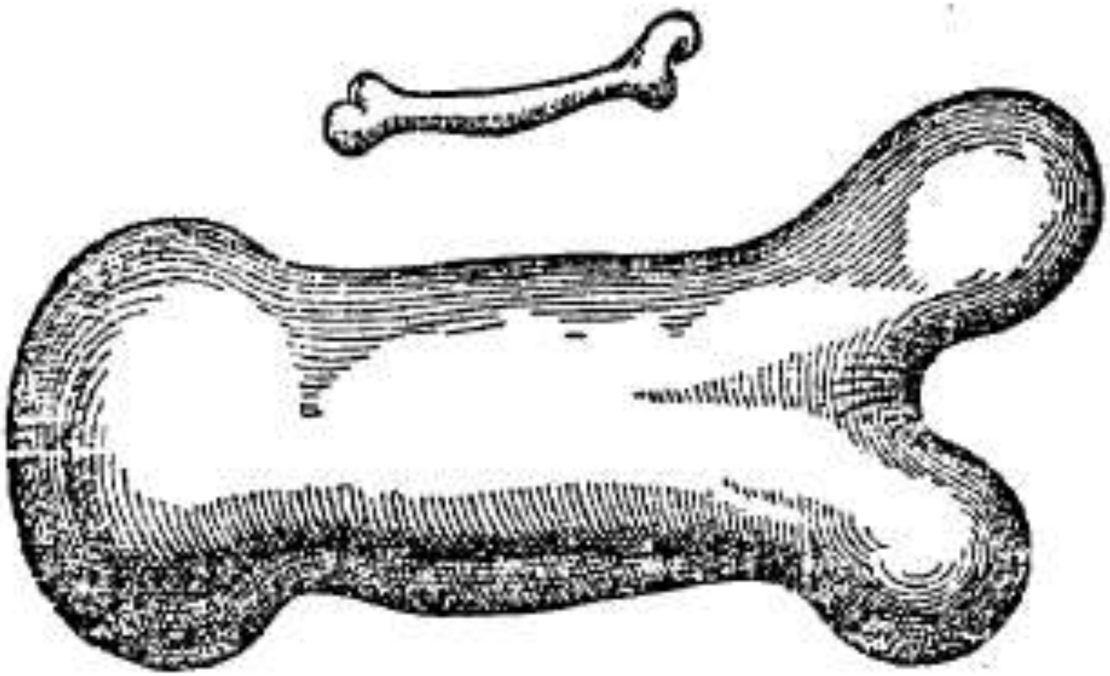
The word “shape” is very commonly used in everyday language, usually referring to the geometry of an object.





School performance test

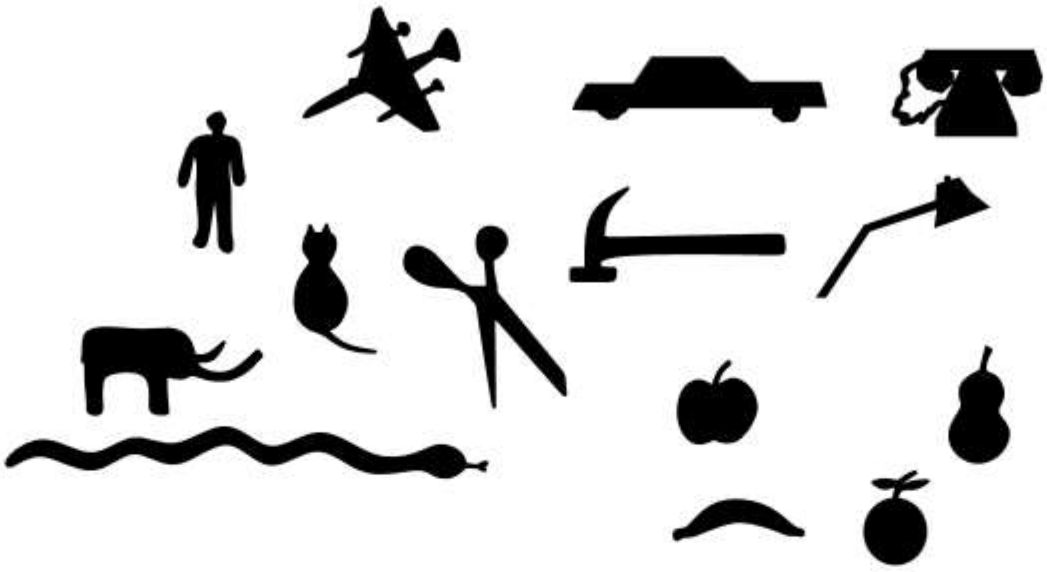
Concept of shape is not new



From Galileo (1638) illustrating the differences in shapes of the bones of small and large animals.

Shape and Human Vision

"Our Visual world contains a vast arrangement of objects, yet we are amazingly robust in recognizing them. This includes objects projected from novel viewpoints, or partially occluded objects. We are even able to describe totally unfamiliar objects, or to recognize unexpected ones out of context."

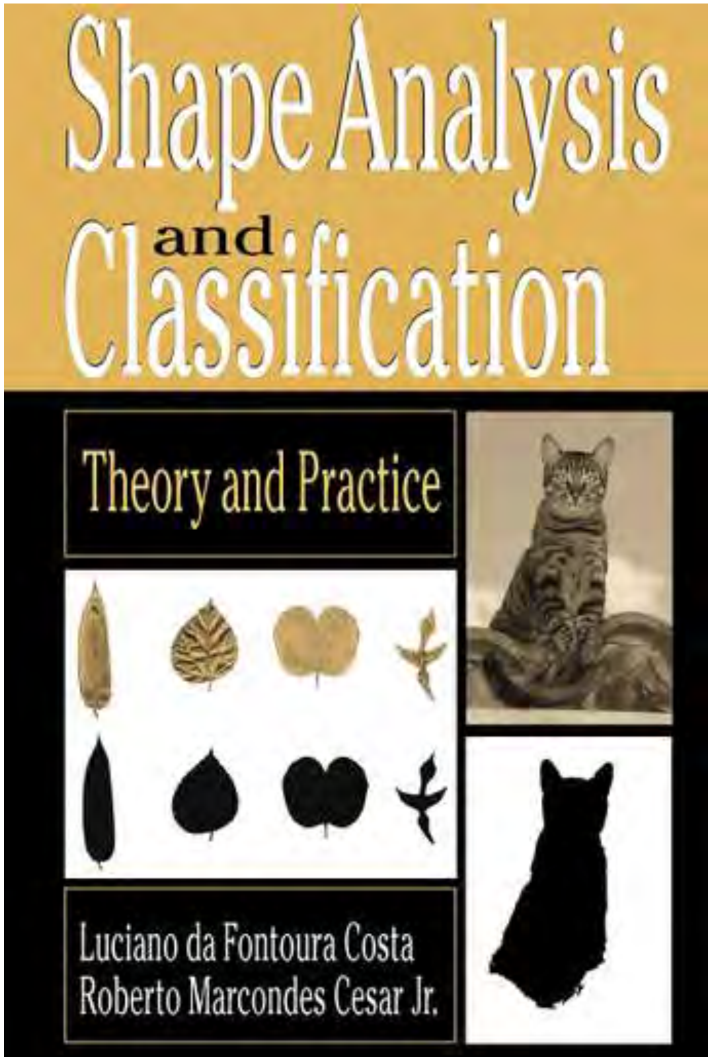


What aspect of the geometry should be computed to allow robust recognition?

Formal Definition? Theory of Shape?

Kimia, Tannenbaum, Zucker, IJCV 1995

Shape Classification



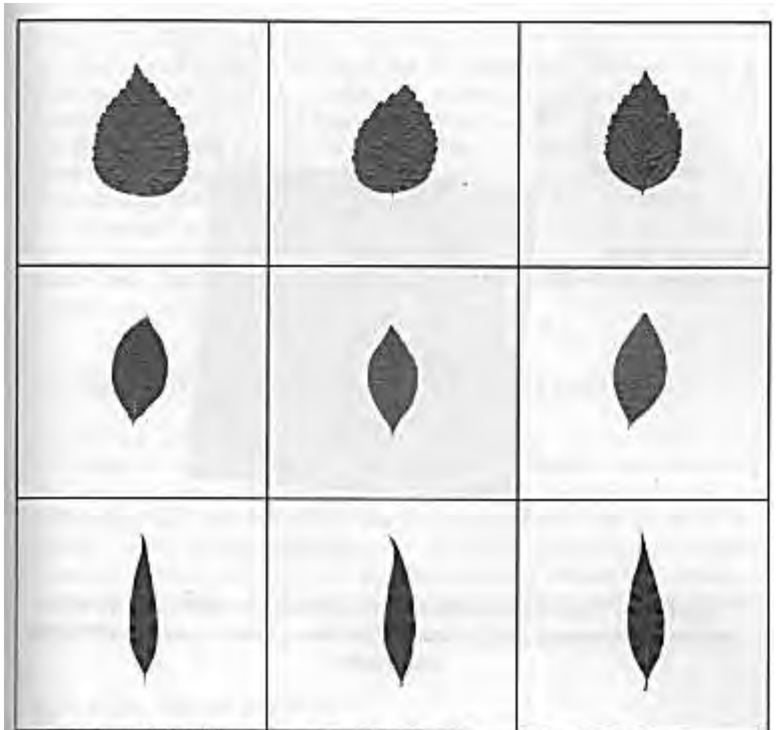


Figure 8.23: *Three examples of each of the considered leaf classes.*

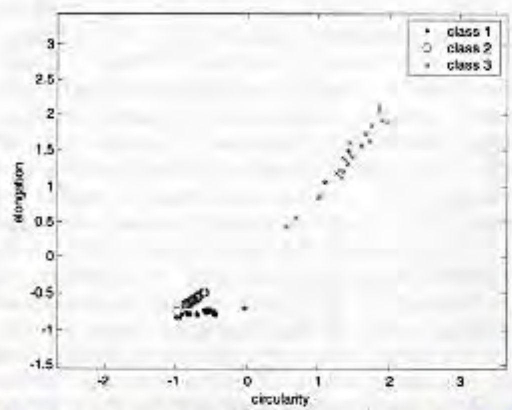
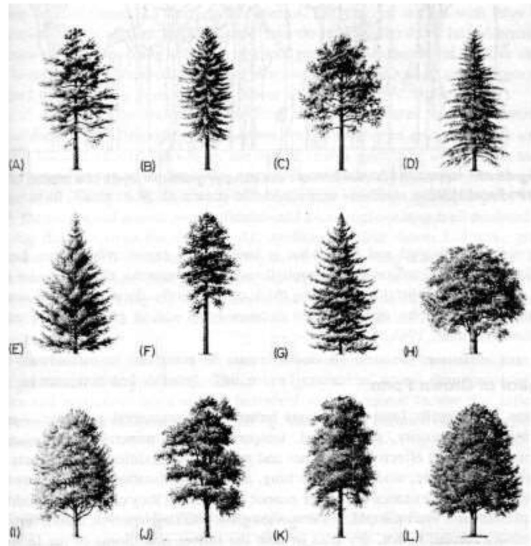
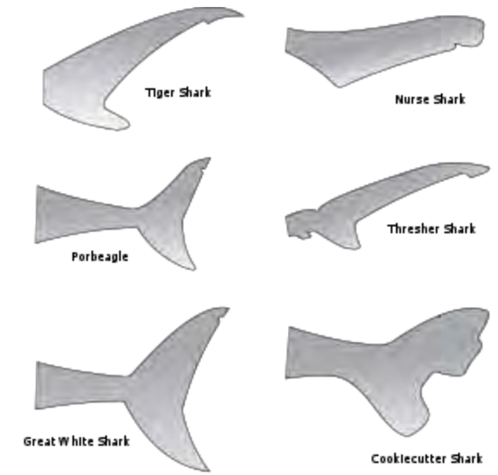


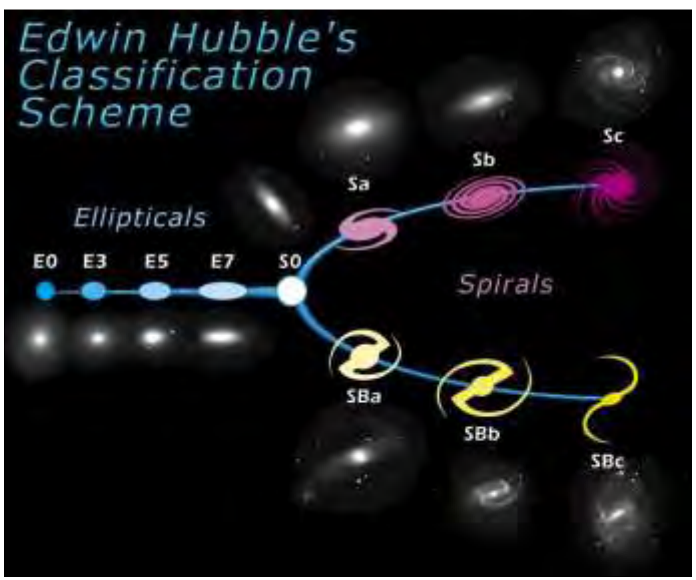
Figure 8.24: *The two-dimensional feature space defined by the circularity and elongation measures, after normal transformation of the feature values. Each class is represented in terms of the 25 observations.*

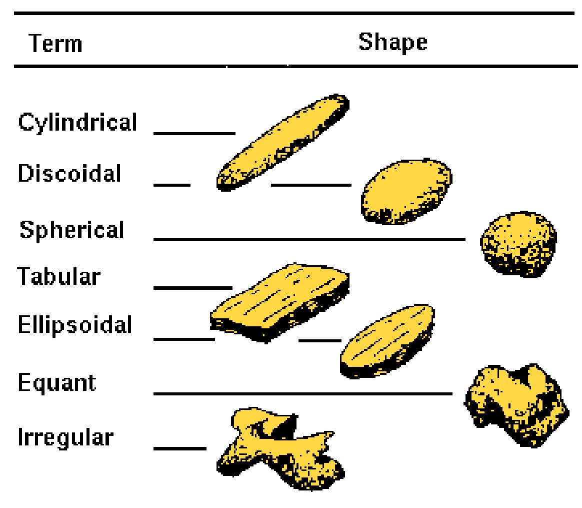
Application Domains



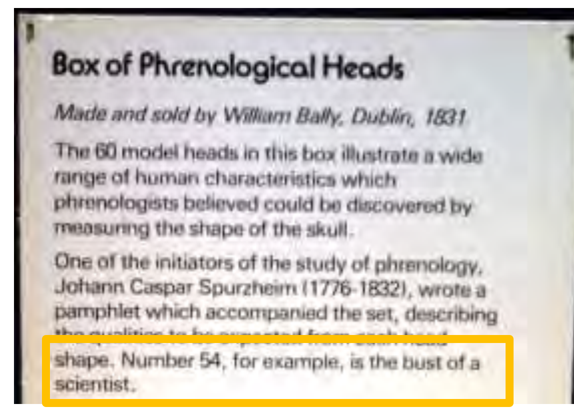


Shark Tails
The Diversity of Form and Function





Shape Statistics: Variability



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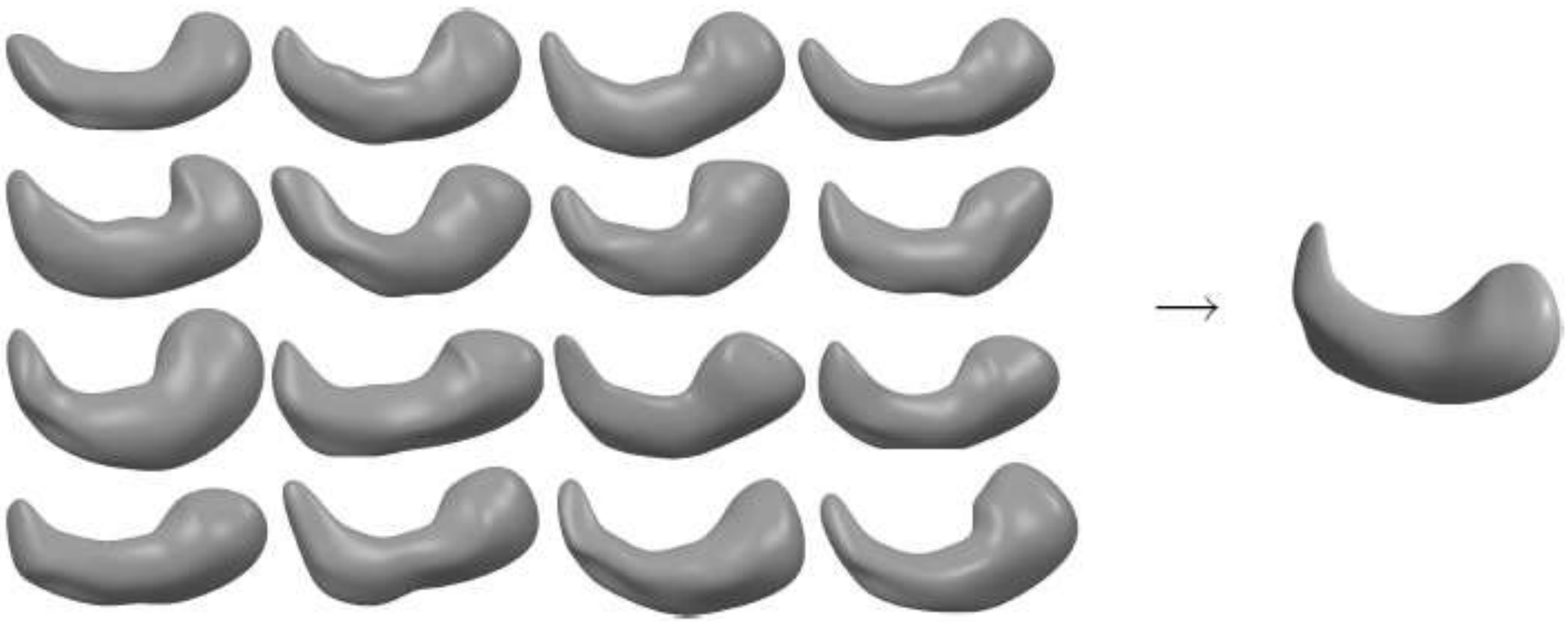
Shape Statistics: Average? Variability?



Shape Metamorphosis

Janine Antoni: Two self portrait busts, 1993 (SFMOMA)

http://www.artnews.com/2013/02/21/chocolate-self-portraits-by-janine-antoni-and-dieter-rot/

Shape Metamorphosis

Janine Antoni, Lick and Lather, 1993-1994 (SFMOMA)

Two self-portrait busts: one chocolate and one soap.
Defacing: Washing soap head in bathtub -> erosion, fetal features, like MCF
Licking chocolate head -> altering features

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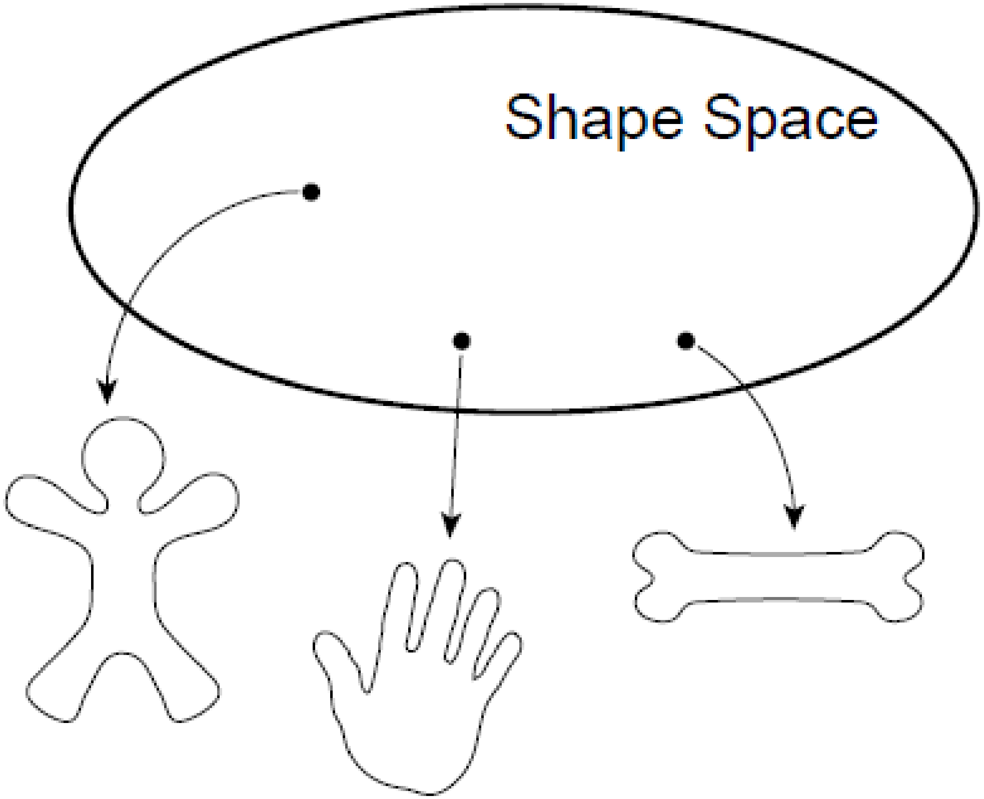
Shape Space



A shape is a point in a high-dimensional, nonlinear shape space.

Pedagogy: Goals

- Terms
  - “Shape Representations”, “Shape Analysis”, “Shape Space”
  - “Kendal Shape Space”
  - “SSM”, “PCA”, “PGA”, “ASM”, “AAM”
  - “Diffeomorphisms”, “Ambient Space”
- Concepts
  - Correspondences/Landmarks in 2-D and 3-D
  - Generation of Statistical Shape Models
  - Use of SSMs for Deformable Model Segmentation
  - Correspondence-Free Shape Analysis
  - Statistics of Deformations of Ambient Space: Deformetrics

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What is Shape?

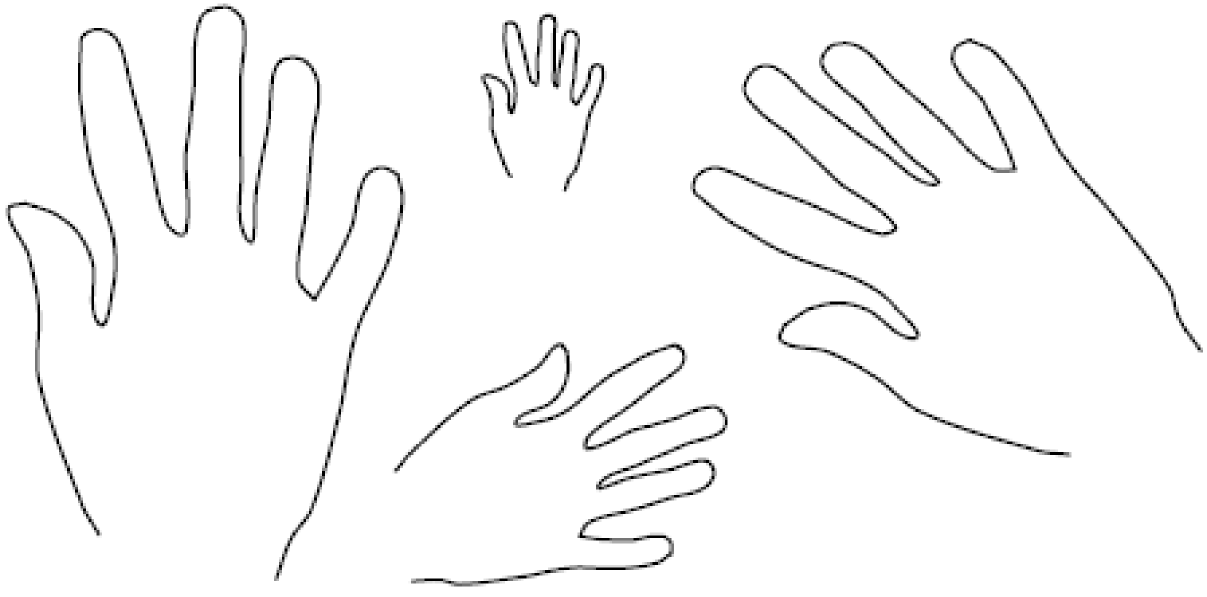
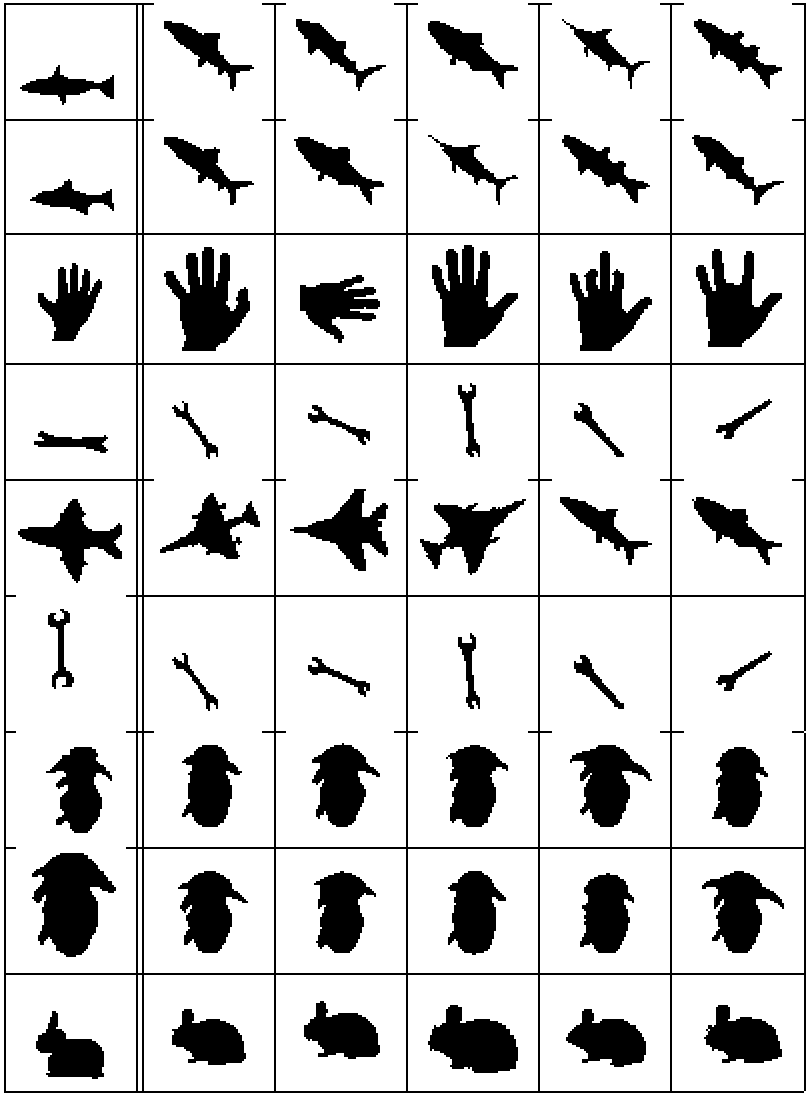


Figure 1: Four copies of the same shape, but under different Euclidean transformations.

Shape is the geometry of an object modulo position, orientation, and size.

Kendall '77, Dryden, Mardia
From: Stegmann and Gomez, [Kendall, 1977]

Shape Definition



Dryden/Mardia, (Kendall 1977):

Shape is all the geometrical information that remains when location, scale and rotational effects are filtered out from an object.

Image: Sebastian and Kimia 2005

Shape Equivalences

Two geometry representations, x_1, x_2, are **equivalent** if they are just a translation, rotation, scaling of each other:

$$x_2 = \lambda R \cdot x_1 + v,$$

where λ is a scaling, *R* is a rotation, and v is a translation.

In notation: $x_1 \sim x_2$

Equivalence Classes

The relationship $x_1 \sim x_2$ is an **equivalence relationship**:

- Reflexive: $x_1 \sim x_1$
- Symmetric: $x_1 \sim x_2$ implies $x_2 \sim x_1$
- Transitive: $x_1 \sim x_2$ and $x_2 \sim x_3$ imply $x_1 \sim x_3$

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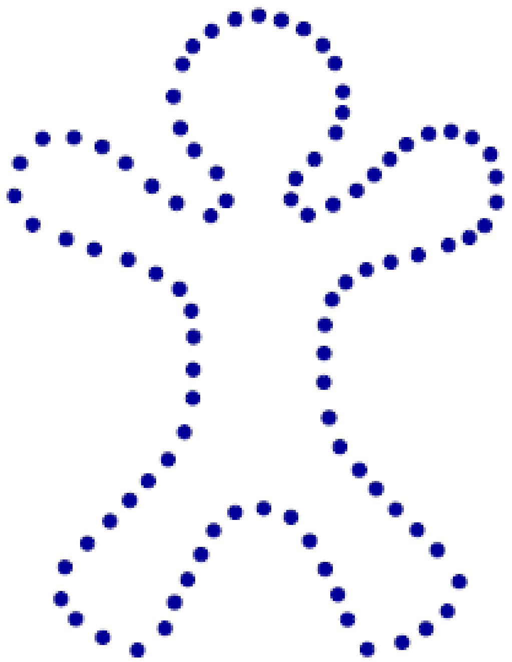
- The set of all equivalence classes is our **shape space**.

Kendall's Shape Space

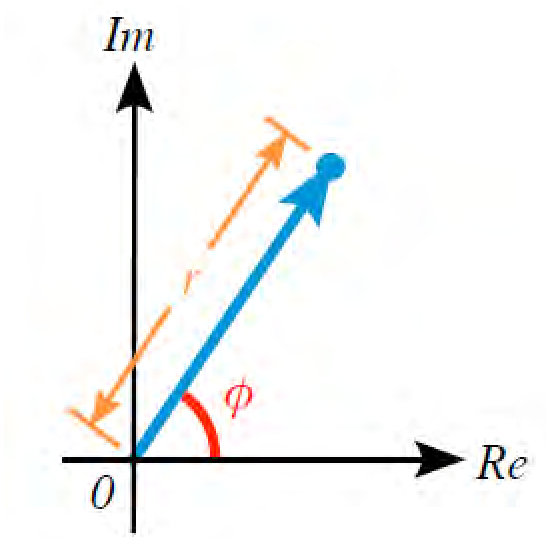


- Define object with k points.
- Represent as a vector in $\mathbb{R}^{2k}$.
- Remove translation, rotation, and scale.
- End up with complex projective space, $\mathbb{CP}^{k-2}$.

Constructing Kendall's Shape Space

- Consider planar landmarks to be points in the complex plane.
- An object is then a point $(z_1, z_2, \cdots, z_k) \in \mathbb{C}^k$.
- Removing **translation** leaves us with $\mathbb{C}^{k-1}$.
- How to remove **scaling** and **rotation**?

Scaling and Rotation in the Complex Plane



Recall a complex number can be written as $z = re^{i\phi}$, with modulus r and argument ϕ.

Complex Multiplication:

$$se^{i\theta} * re^{i\phi} = (sr)e^{i(\theta+\phi)}$$

Multiplication of z by a complex number $se^{i\theta}$ is equivalent to scaling by s and rotation by θ.

Removing Scale and Rotation

Multiplying a centered point set, $\boldsymbol{z} = (z_1, z_2, \cdots, z_{k-1})$, by a constant $w \in \mathbb{C}$, just rotates and scales it.

Thus the shape of $\boldsymbol{z}$ is an equivalence class:

$$[\boldsymbol{z}] = \{(wz_1, wz_2, \cdots, wz_{k-1}) : \forall w \in \mathbb{C}\}$$

This gives complex projective space $\mathbb{CP}^{k-2}$.

(Note: centering 1DOF, rotation 2DOF (1 in complex space) $\longrightarrow \mathbb{CP}^{k-2}$)

Non-Euclidean Shape Space

- Shape Space = complex projective space $\mathbb{CP}^{k-2}$.
- Shape distance between two objects z, w:

$$\rho = \arccos \frac{\sum (z_j - \bar{z})^* (w_j - \bar{w})}{\left(\sum |z_j - \bar{z}|^2 \sum |w_j - \bar{w}|^2\right)^{1/2}}$$

Shape Distances …

- Partial Procrustes distance:

$$d_P(X_1, X_2) = \inf_{\Gamma \in SO(m)} \|Z_2 - Z_1 \Gamma\|,$$

- Riemannian metric:

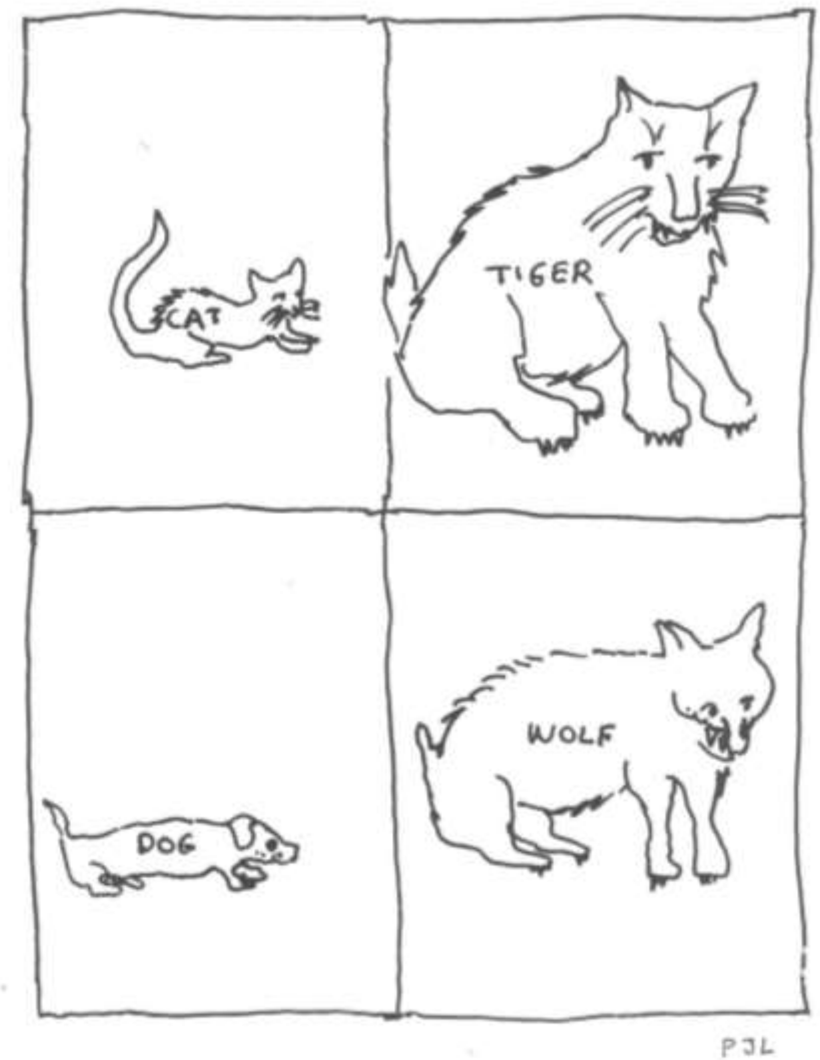
$$\rho(X_1, X_2) = 2 \arcsin(d/2), \quad (0 \leq \rho \leq \pi/2).$$

- Full Procrustes distance:

$$d_F(X_1, X_2) = \inf_{r>0, \Gamma} \|Z_2 - r Z_1 \Gamma\| = \sin \rho(X_1, X_2)$$

Ian Dryden: Dryden-Stat-Shape-analysis-2000.pdf

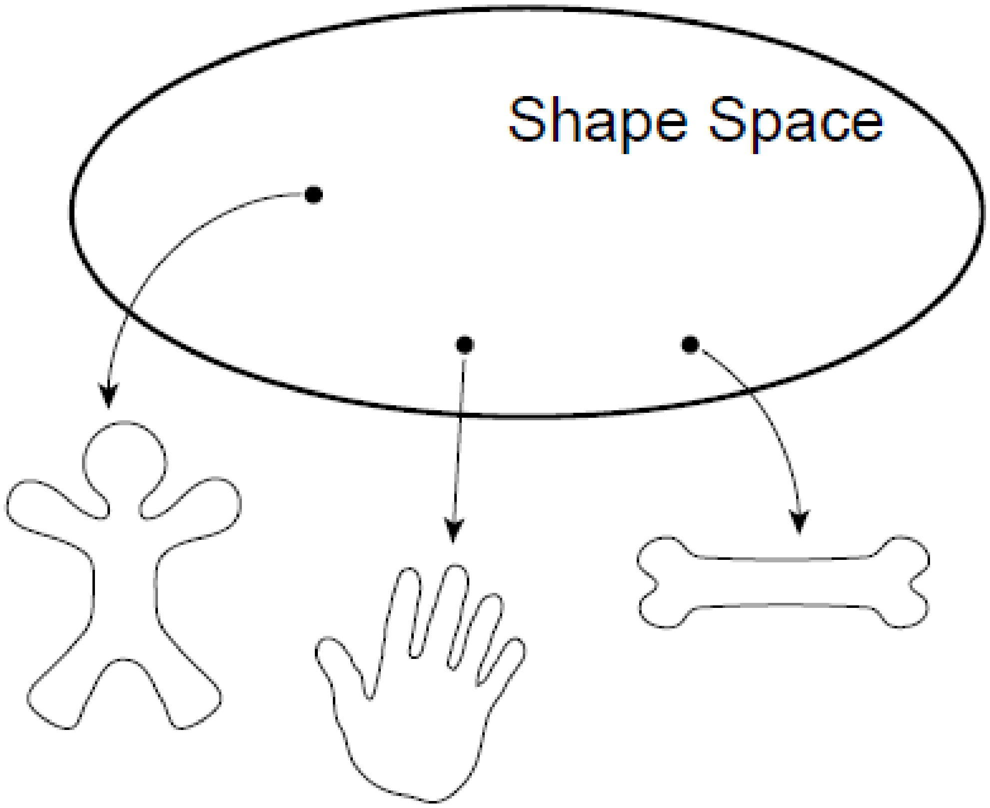
The Problem of Size and Shape



Dryden/Mardia (Kendall 1977): *(Sometimes we are also interested in retaining scale information as well as shape)* →

Size-and-Shape is all the geometrical information that remains when location and rotational effects are filtered out from an object.

Shape Analysis



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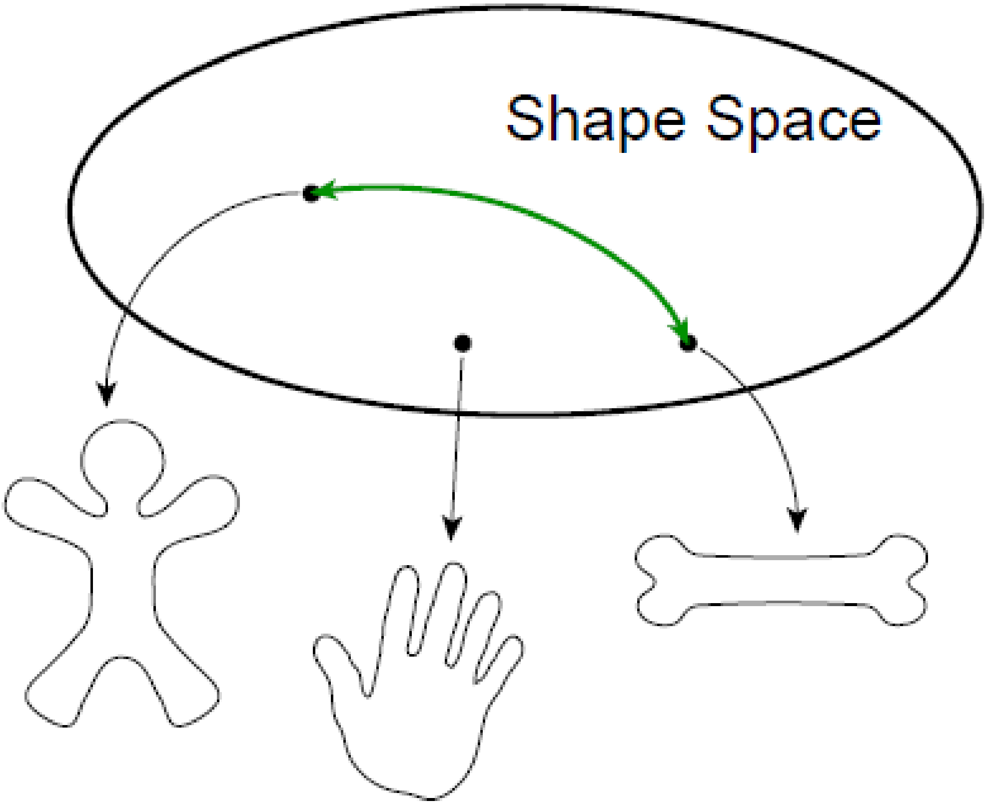
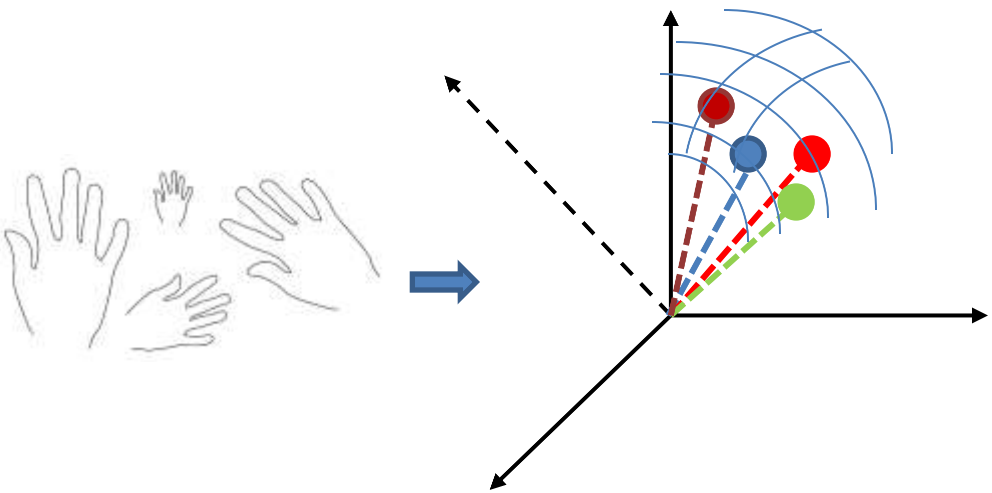
Shape Analysis



A metric space structure provides a comparison between two shapes.

Shape Space for Object Class

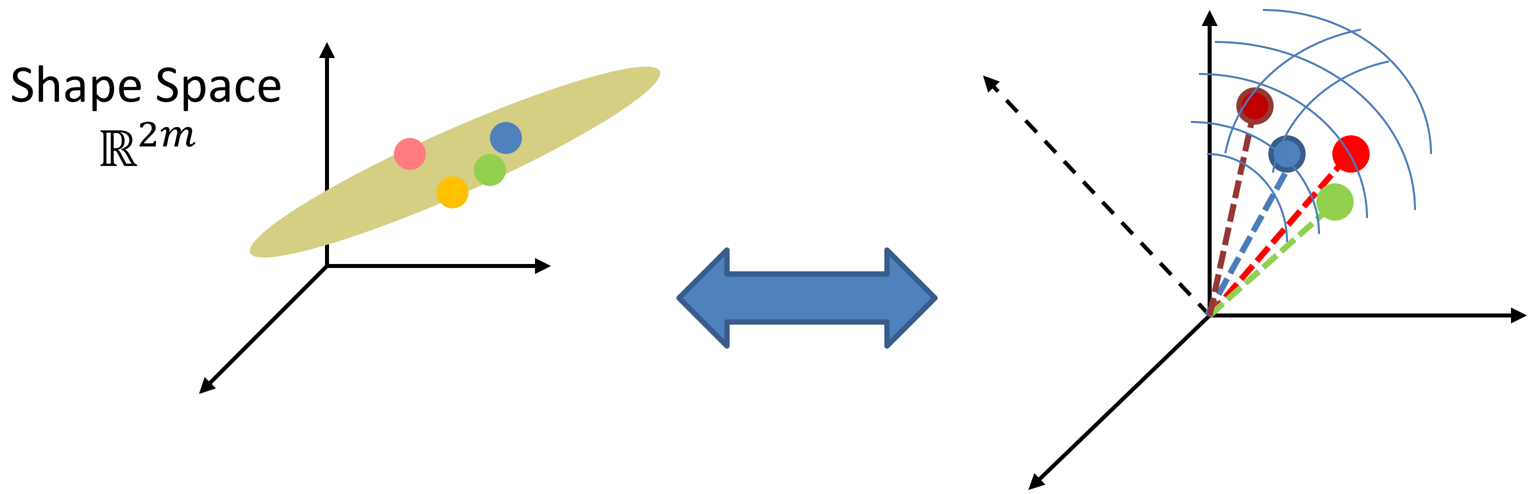


- k landmarks in n Euclidean dimensions: kn-dim space
- Procrustes alignment: Shape vectors of length dimensionality kn normalized for size $\rightarrow L2$ norm
- Vectors lie on subpart of a $\boldsymbol{kn}$-**dimensional hyper sphere**

- **Linear methods** are nice, but shape space is curved surface: Hyper sphere.
- Standard statistics (μ, Σ) not build for hyperspheres.
- **Tangent-space projection**: Modify shape vectors to form hyper plane.
- Use **Euclidean distance** in this plane rather than true geodesic distance.

Stegmann and Gomez, 2002

Structure of Shape Space



Assumption SSM: Multivariate Gaussian distribution, **linear** stats

Kendal Shape Space: Part of Hypersphere, **curved** manifold

Shape Space: Tangent-Space Projection

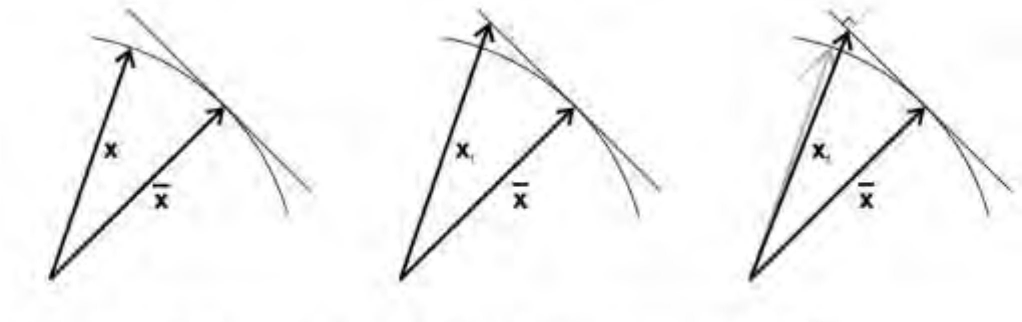


Figure 2: Left: One shape vector drawn from a population, $\mathbf{x}$ and the mean, $\overline{\mathbf{x}}$. Middle: Tangent space projection by scaling. Right: Tangent space projection by scaling and shape modification.

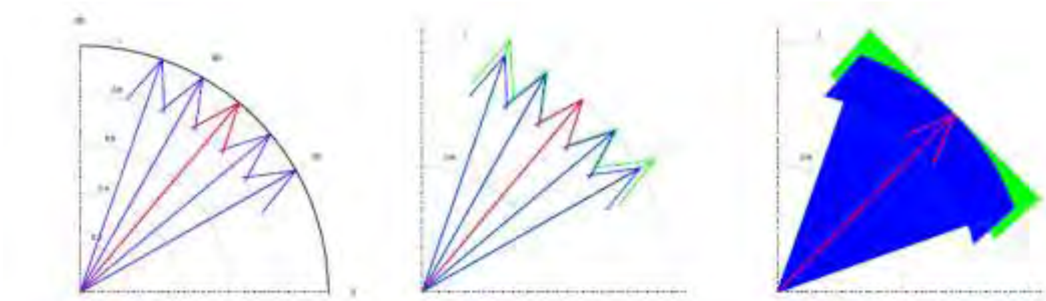
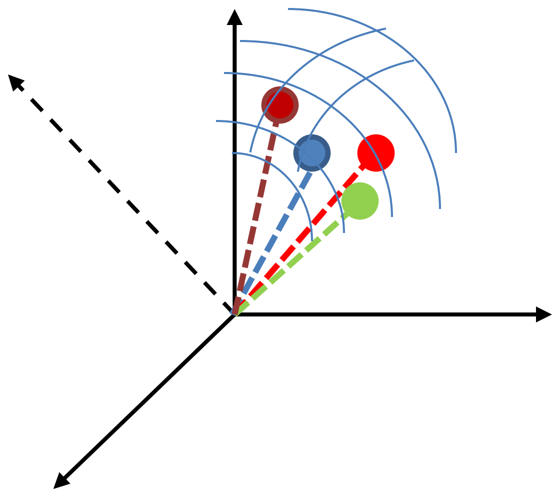


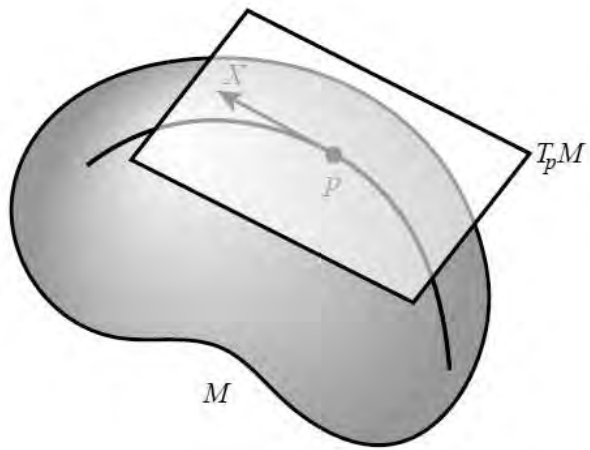
Figure 3: Left: A planar projection of four aligned shapes (mean shape shown in red). Middle: Same as left with tangent space projection (shown in green). Right: Same as middle on four hundred vectors.

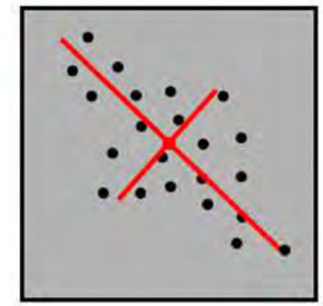
- Project shape vectors to tangent space.
- Apply standard statistics (μ, Σ).
- Shown to be good approximation (not much difference) in case of small shape variability.

Stegmann and Gomez, 2002

Shape Space: Tangent-Space Projection





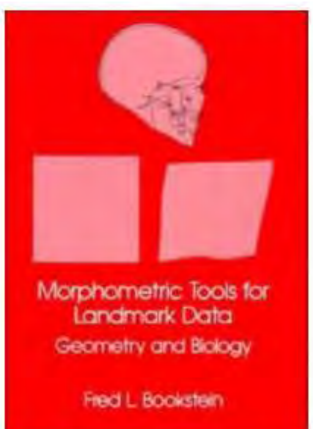


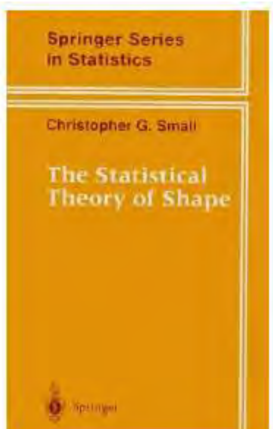
Linear Statistics
Principal Components Analysis
(PCA)

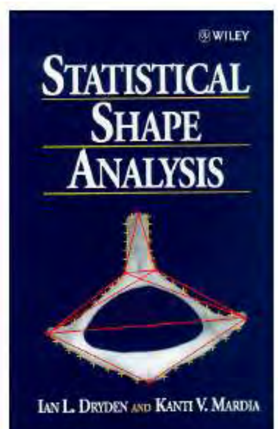
Calculate tangent space, projection to tangent space, linear statistics.

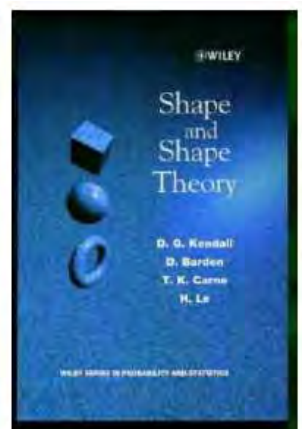
Where to Learn More

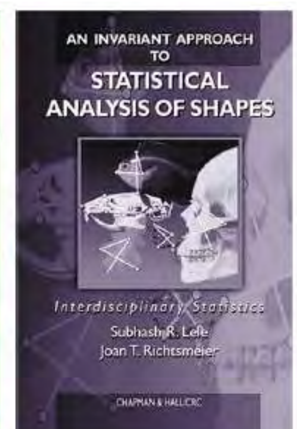
- **Pioneers: Fred Bookstein and David Kendall.**
- Bookstein (1991, Cambridge).
- Kendall, Barden and Carne, Shape and Shape Theory, Wiley, 1999.
- Dryden and Mardia, Statistical Shape Analysis, Wiley, 1998.
- Small, The Statistical Theory of Shape, Springer-Verlag, 1996.
- Grenander, HISTORY AS POINTS AND LINES, 1998-2003
- Lele and Richstmeier (2001, Chapman and Hall).
- Krim and Yezzi, Statistics and Analysis of Shapes, Birkhauser, 2006.



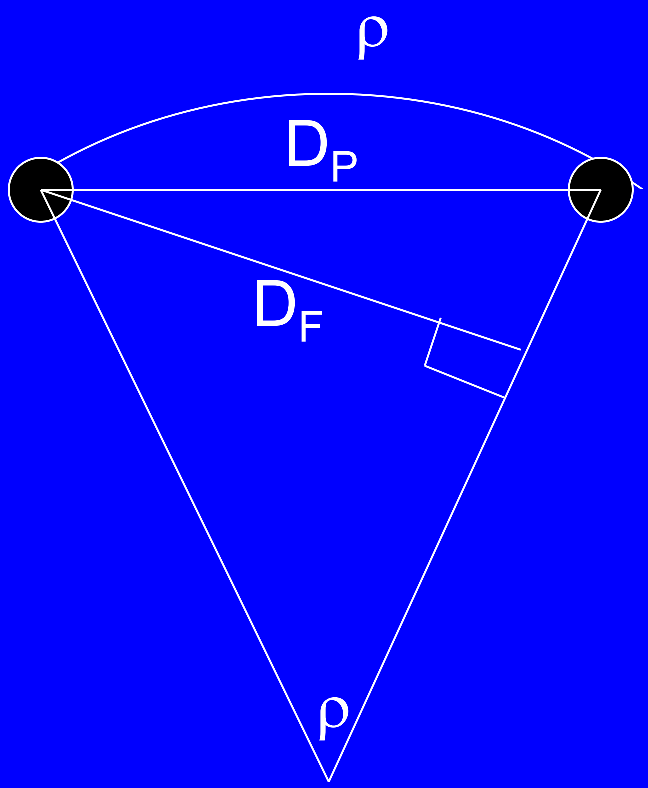








Given two points on the hypersphere, we can draw the plane containing these points and the origin.



Procrustes Distances is ρ.

$D_P = 2 \sin(\rho/2)$

$D_F = \sin \rho$.

- These are all monotonic in ρ. So the same choice of rotation minimizes all three.

- D_F is easy to compute, others are easy to compute from D_F.

Why Procrustes Distance?

- Procrustes distance is most natural. Our intuition is that given two objects, we can produce a sequence of intermediate objects on a 'straight line' between them, so the distance between the two objects is the sum of the distances between intermediate objects. This requires a geodesic.

Tangent Space

- Can compute a hyperplane tangent to the hypersphere at a point in preshape space.
- Project all points onto that plane.
- All distances Euclidean. Average shape easy to find.
- This is reasonable when all shapes similar.
- In this case, all distances are similar too.
  - Note that when ρ is small, ρ, $2\sin(\rho/2)$, $\sin(\rho)$ are all similar.

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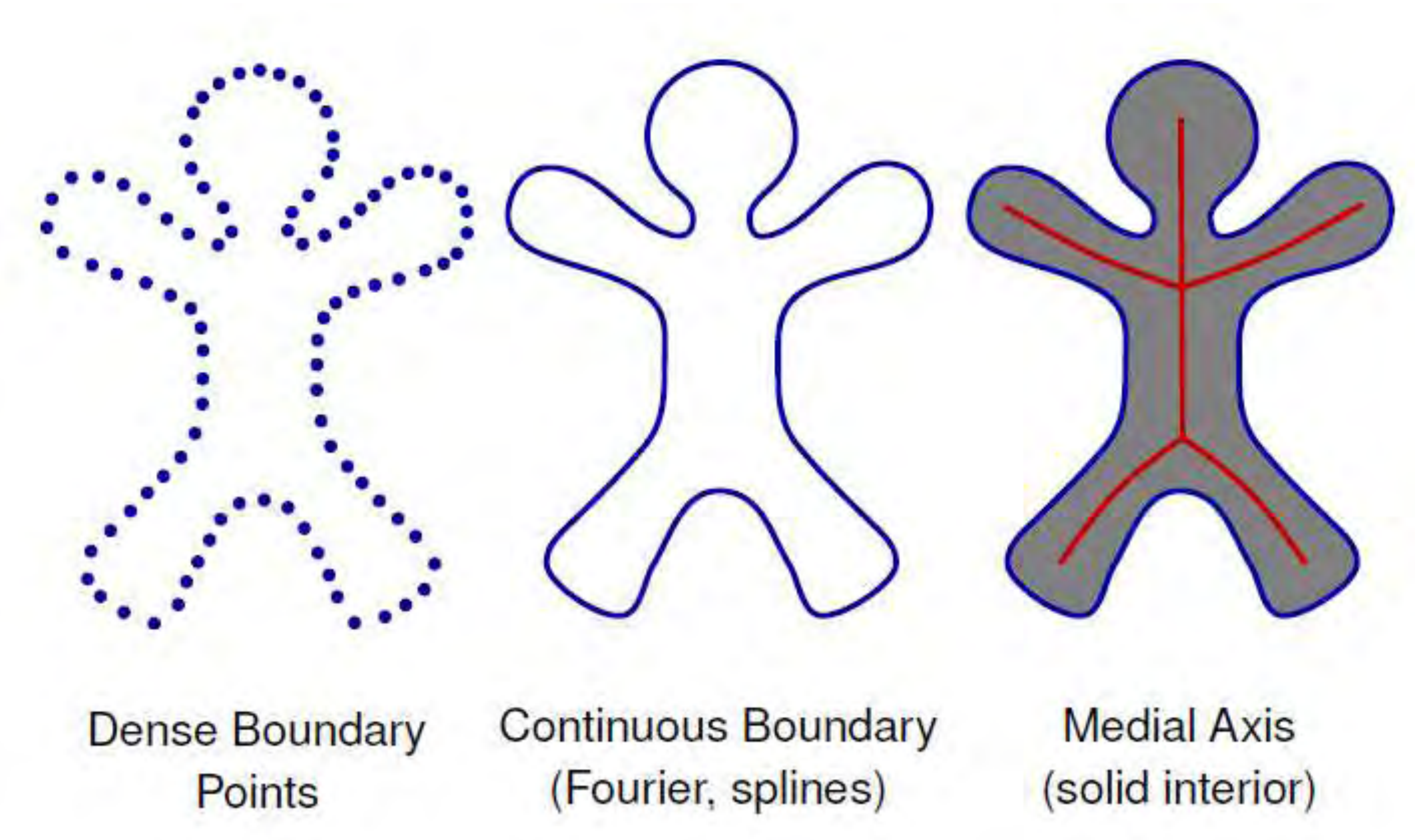
  - Ambient Space Deformations via Diffeomorphisms
  - Statistics of Deformations of Ambient Space

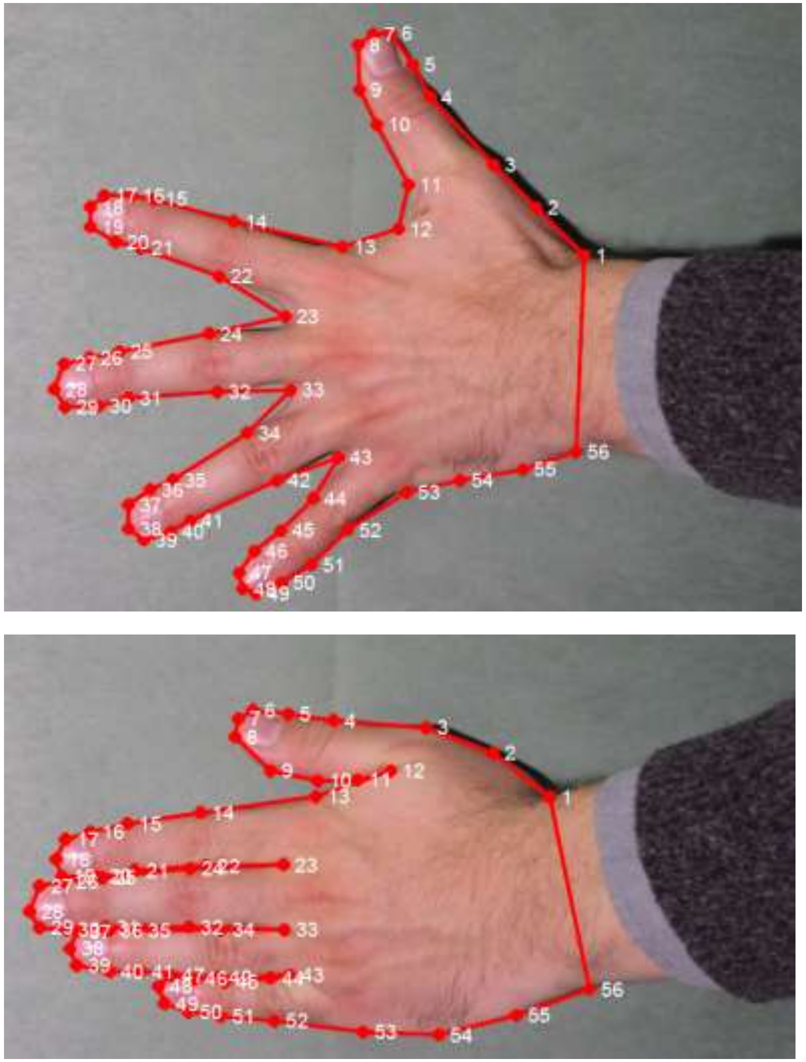
Geometry Representations



Geometry Representations

- Landmarks (key identifiable points)
- Boundary models (points, curves, surfaces, level sets)
- Interior models (medial, solid mesh)
- Transformation models (splines, diffeomorphisms)

Boundary via Landmarks



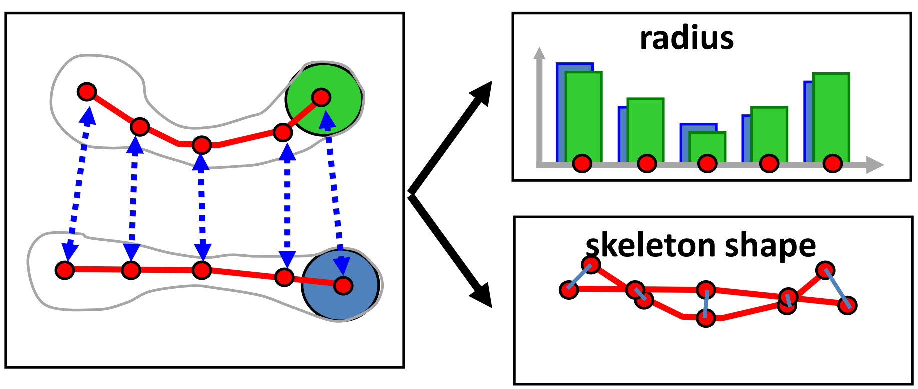
Stegmann and Gomez, 2002

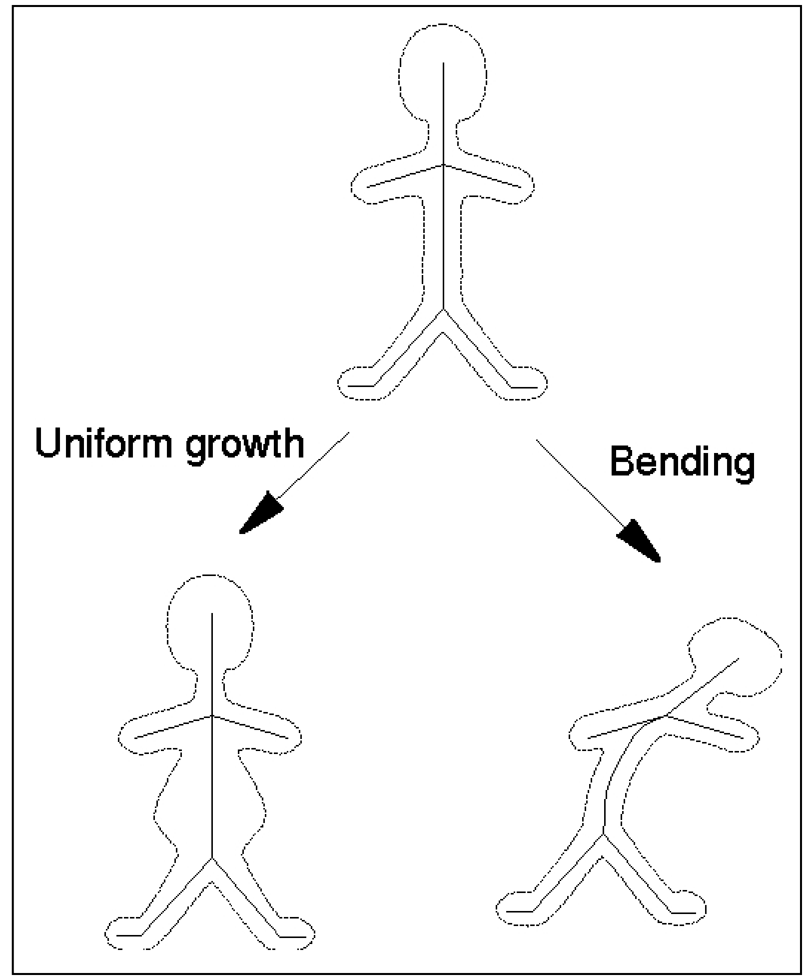
Boundary versus Skeleton

Shape Representation:

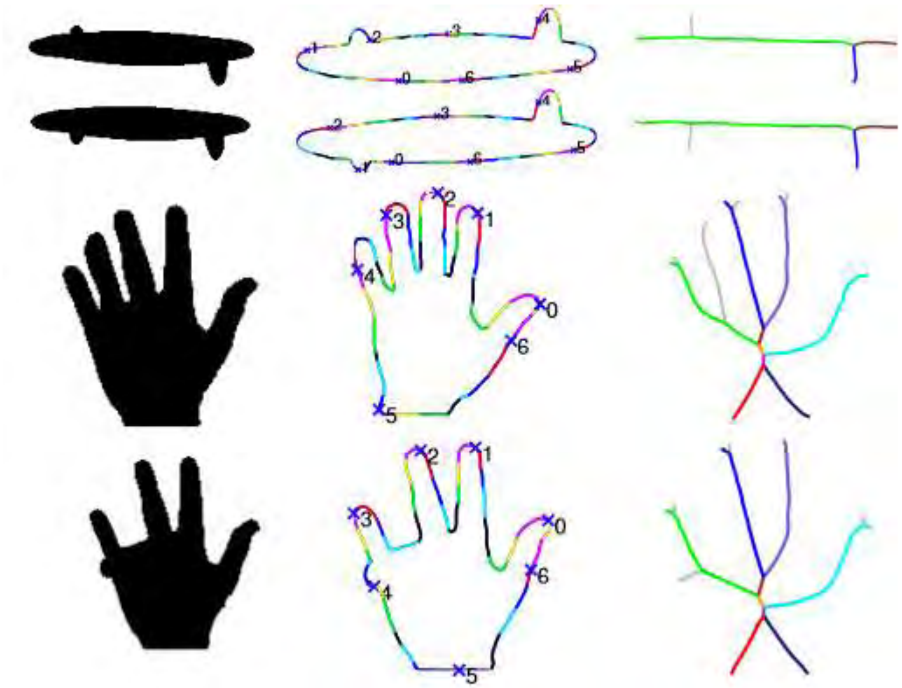
– Contour / Boundary / Surface

– Skeleton (medial model)

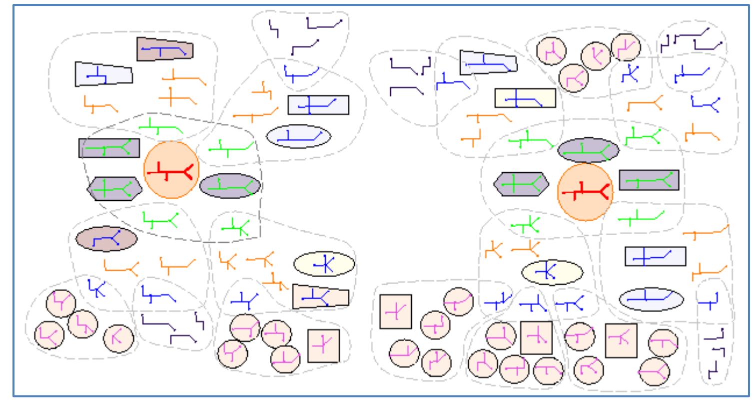




Skeleton Shape Representation



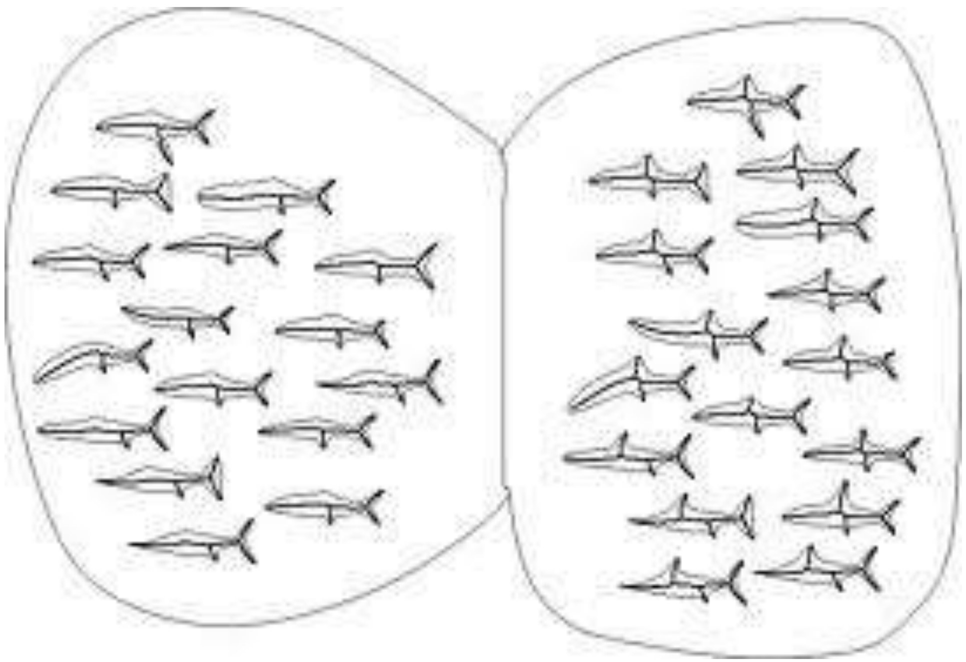
Sensitivity of curve matching to spatial arrangement and how shock graph matching avoids the problem.



Shock Grammar, Symmetry Maps and Transforms For Perceptual Grouping and Object Recognition, Benjamin B. Kimia, Brown

Sebastian and Kimia, Signal Processing, 2005

Shock Graph: Shape Transformation



Invariance of shock graph to flexibly deformable objects.

Matching dog to cat via shock graph editing

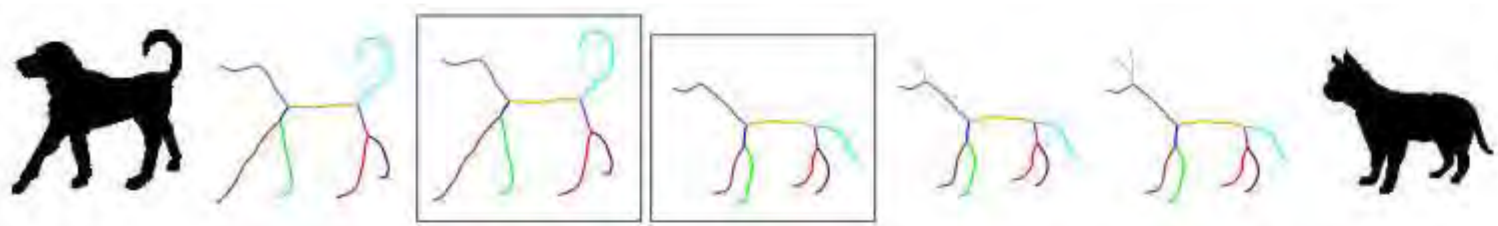


Figure 5: This figure from [43] intermediate shock graphs resulting from applying the edits in the optimal edit sequence for matching a cat and dog. The boxed shock graphs have the same topology. The distance between the shapes is the sum of all edit costs.

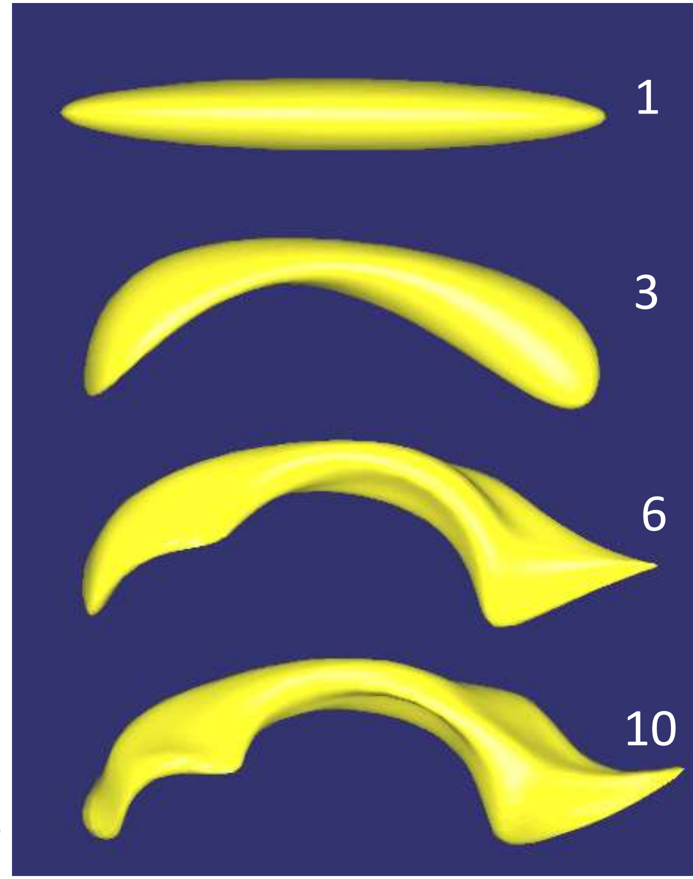
Sebastian and Kimia, Signal Processing, 2005

Spherical Harmonics (SPHARM)

1. Extract voxel surface
2. Area preserving parameterization
3. First order ellipsoid alignment
4. Fit SPHARM to coordinates
5. Sample parameterization and reconstruct object

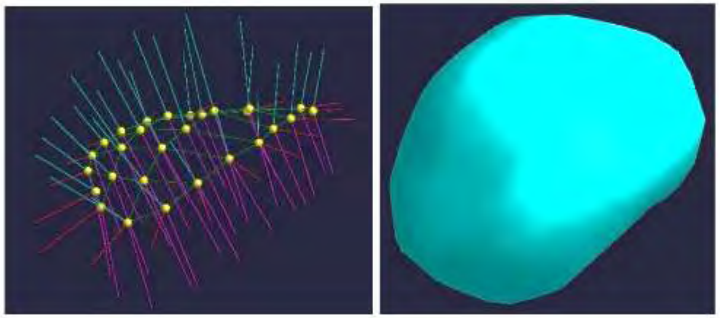
$$\mathbf{r}(\theta,\phi) = \begin{pmatrix} x(\theta,\phi) \\ y(\theta,\phi) \\ z(\theta,\phi) \end{pmatrix} \rightarrow \mathbf{r}(\theta,\phi) = \begin{pmatrix} x(\theta,\phi) \\ y(\theta,\phi) \\ z(\theta,\phi) \end{pmatrix}$$

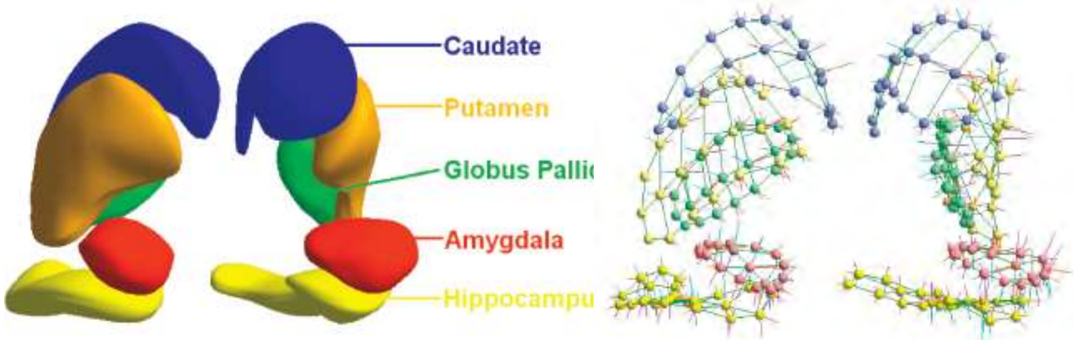
$$\mathbf{r}(\theta,\phi) = \sum_{k=0}^{K} \sum_{m=-k}^{k} \mathbf{c}_k^m \mathbf{Y}_k^m(\theta,\phi) \rightarrow \mathbf{c}_k^m = \begin{pmatrix} c_{xk}^m \\ c_{yk}^m \\ c_{zk}^m \end{pmatrix}$$



Szekely, Kelemen, Brechbuehler, Gerig, MedIA 1996

Medial Axis / Skeletal Representation: Intrinsic Shape Model

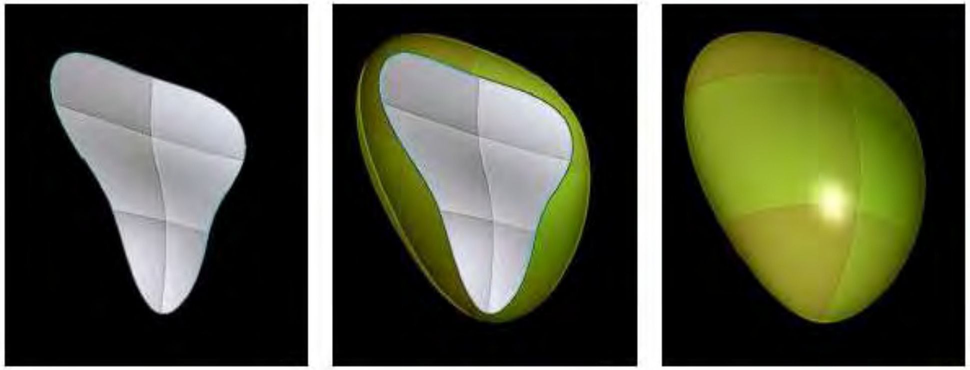




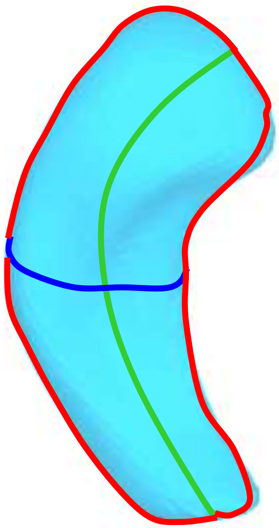
S-rep: Prostate s-rep and implied boundary: Pizer et al. (discrete)

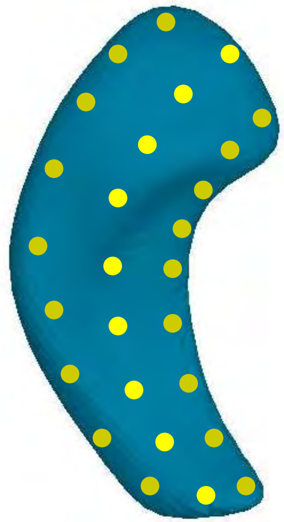
Gorczowski , Pizer, Gerig et al., T-PAMI 2010, Stats on deformations vs. thickness

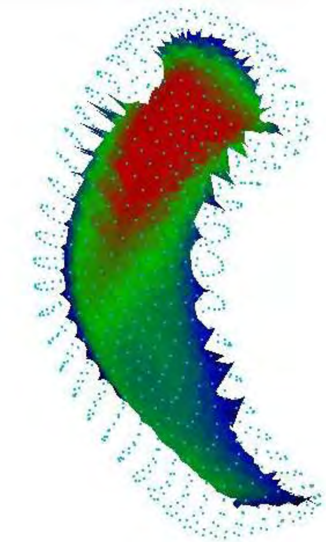
CM-rep: Yushkevich (continuous, parametric) Yushkevich et al., TMI 2006

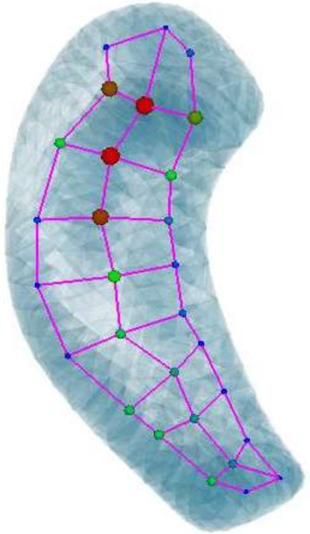


3D Shape Representations









SPHARM

Boundary, fine scale, parametric

$$\mathbf{r}(\theta,\phi)=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\underline{\mathbf{c}}_k^m \mathbf{Y}_k^m(\theta,\phi)$$

PDM

Boundary, fine scale, sampled

Skeleton

Medial, fine scale, sampled

<u>Skeleton from boundary points</u>

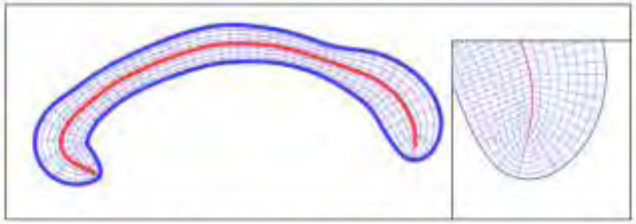
M-rep

Medial, coarse scale, sampled

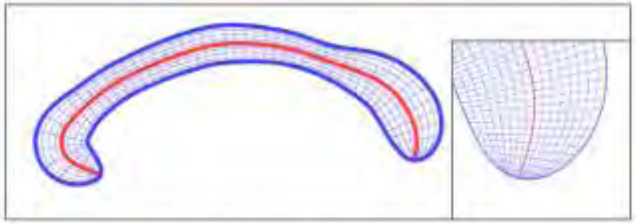
<u>Implied Surface</u>

$$m=\left(\underline{x},r,F,\theta\right)$$

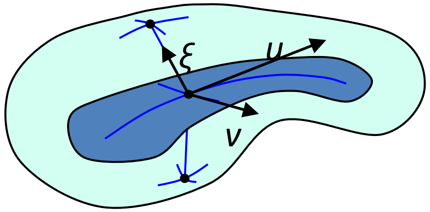
CM-rep

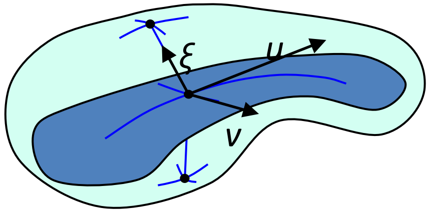


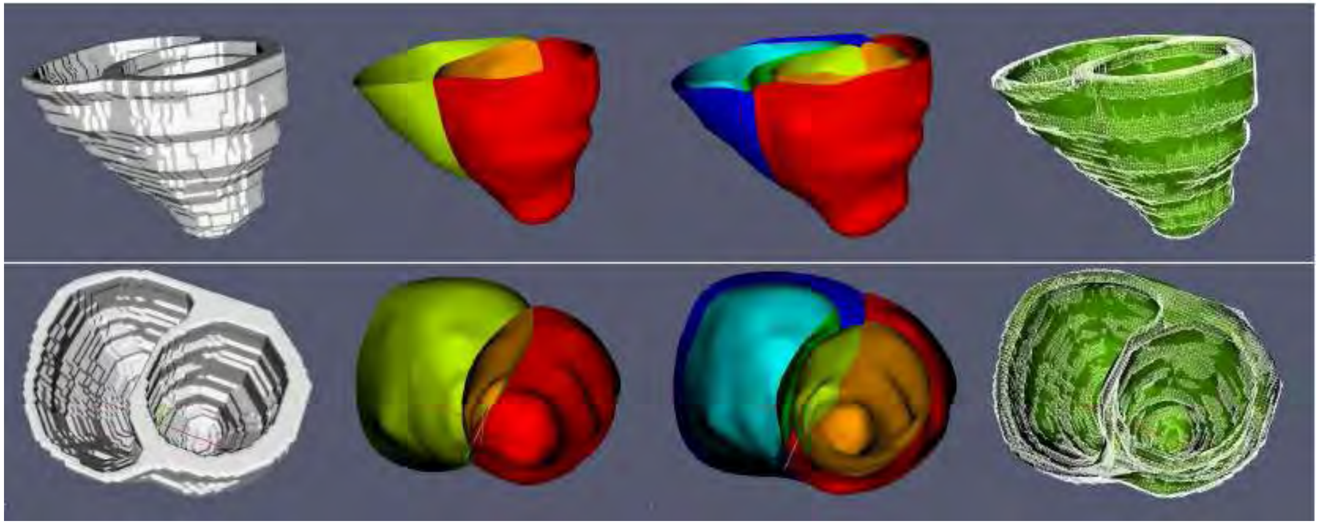
(a) Blum Skeleton Based parametrization



(b) SLS Based parametrization







Sun et al. 2007, IEEE TMI

Level-Set Formulation: Shapes as signed distance functions

- Embed shape contour as 0-level set
- Calculate Euclidean distance transform.
- Contour represented as image with embedded set of signed distance functions.

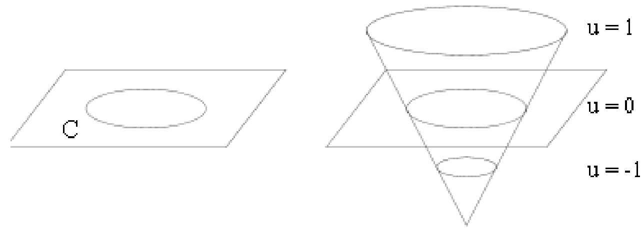
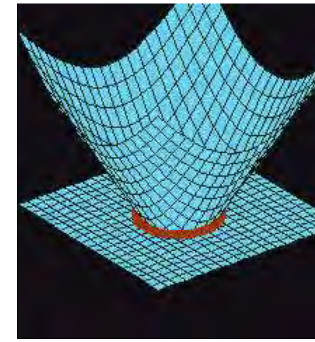


Figure 1. Level sets of an embedding function u for a closed curve C in $\Re^2$.



Leventon, Grimson, Faugeras, CVPR 2000

Volumetric Laplace Spectrum

- “Shape DNA”: Fingerprint, Signature
- Laplace-Beltrami Spectrum
- Global Shape Descriptor
- Voxel object: no registration, no mapping, no re-meshing

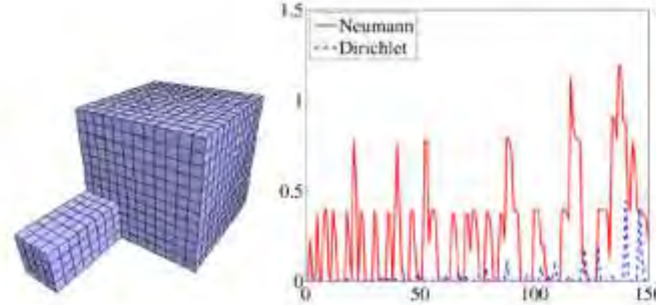


Figure 6.
The first 150 eigenvalues of the cube with tail subtracted from the eigenvalues of the cube for the Dirichlet and Neumann case.

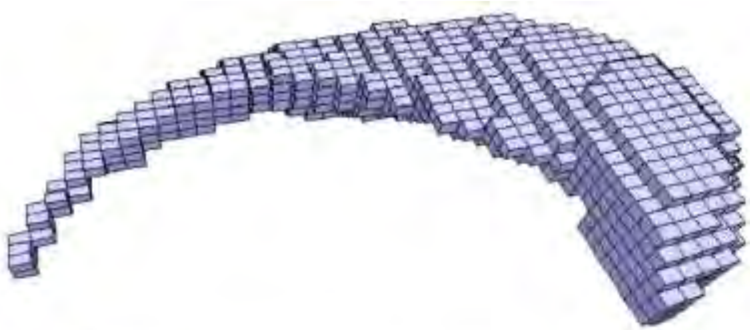


Figure 11.
Example of a caudate shape consisting of voxels

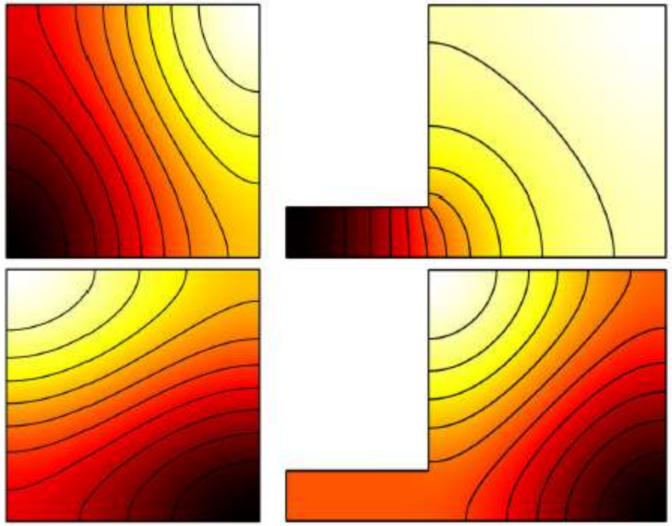
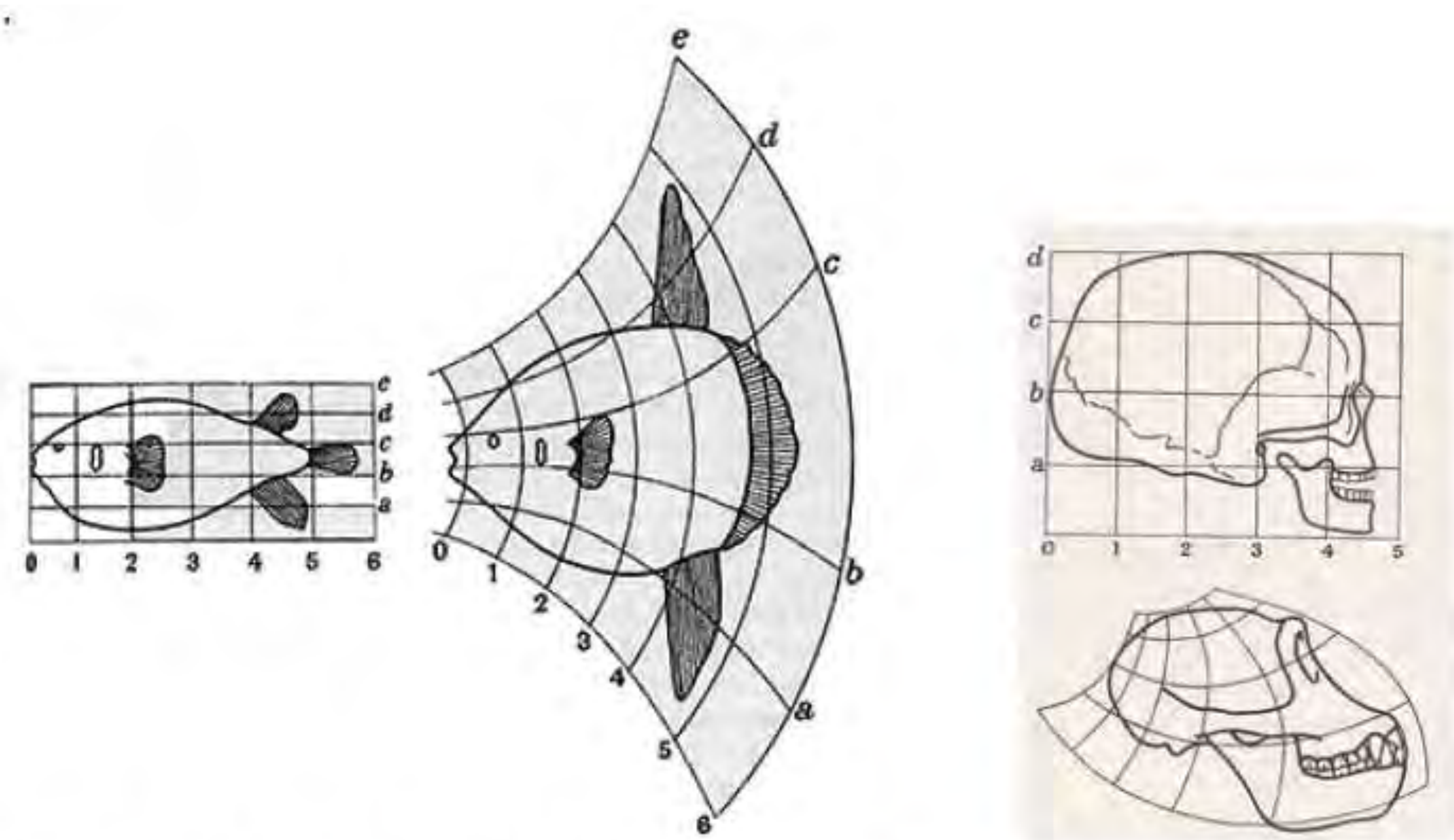


Figure 8.
Neumann Eigenfunctions 2 (top) and 3 (bottom)

Reuter, Niethammer, Bouix, MICCAI 2007

Transformation Models



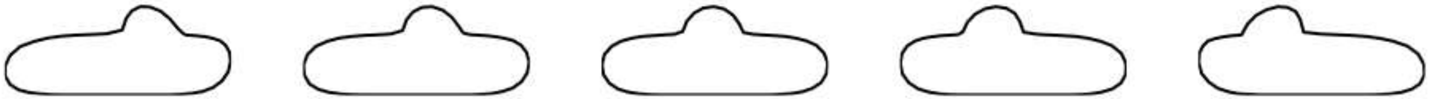
From D'Arcy Thompson, *On Growth and Form*, 1917.

Contents

- What is Shape?
- Geometry Representations
- Kendall Shape Space
  - Statistical Shape Modeling (SSM)
  - Correspondences
  - Active Shape & Appearance Models (ASM, AAM)
- Shape Statistics via Deformations
  - Correspondence-free Mapping & Stats via "currents"
  - Ambient Space Deformations via Diffeomorphisms
  - Statistics of Deformations of Ambient Space

Statistical Shape Analysis

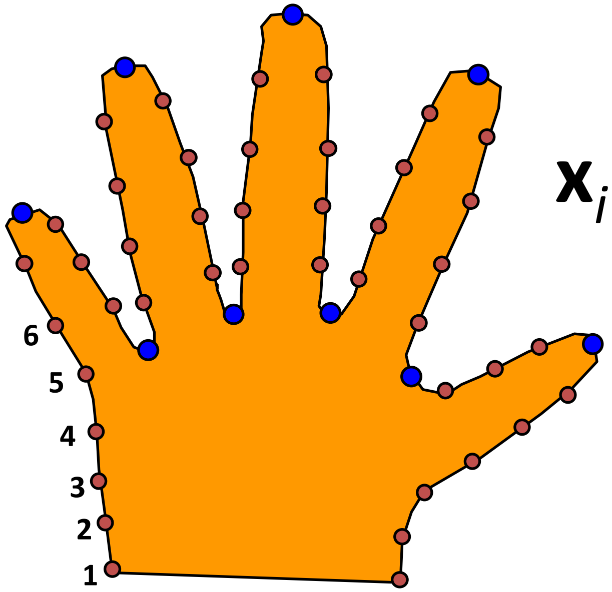
- What is the mean of these shapes?



- Quantify variability
- Quantify individuals relative to population
- Hypothesis testing
- Regression

Modelling Shape

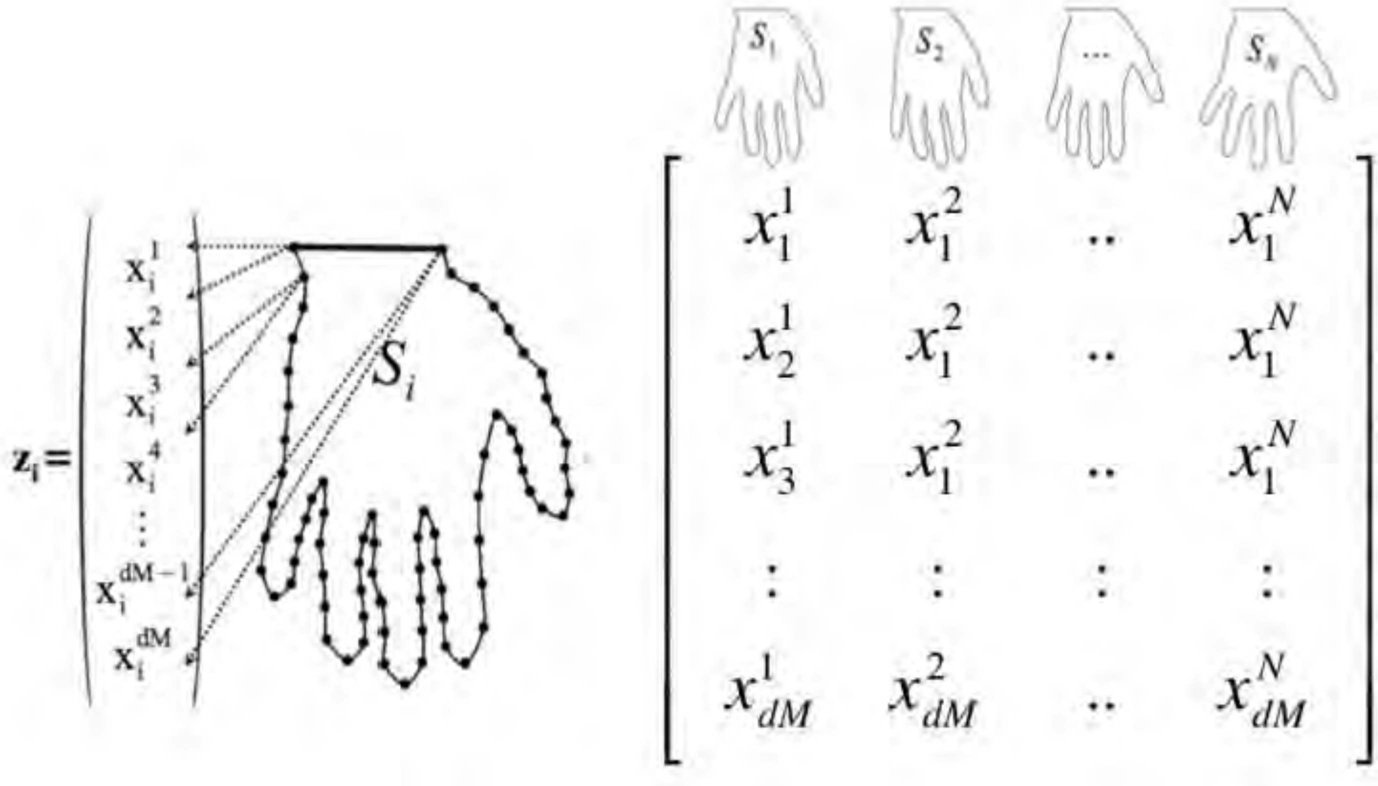
- Define each example using points
- Each (aligned) example is a vector



$$\mathbf{x}_i = \{x_{i1}, y_{i1}, x_{i2}, y_{i2} \ldots x_{in}, y_{in}\}$$

Cootes, Taylor 1993

SSM: Point Distribution Model

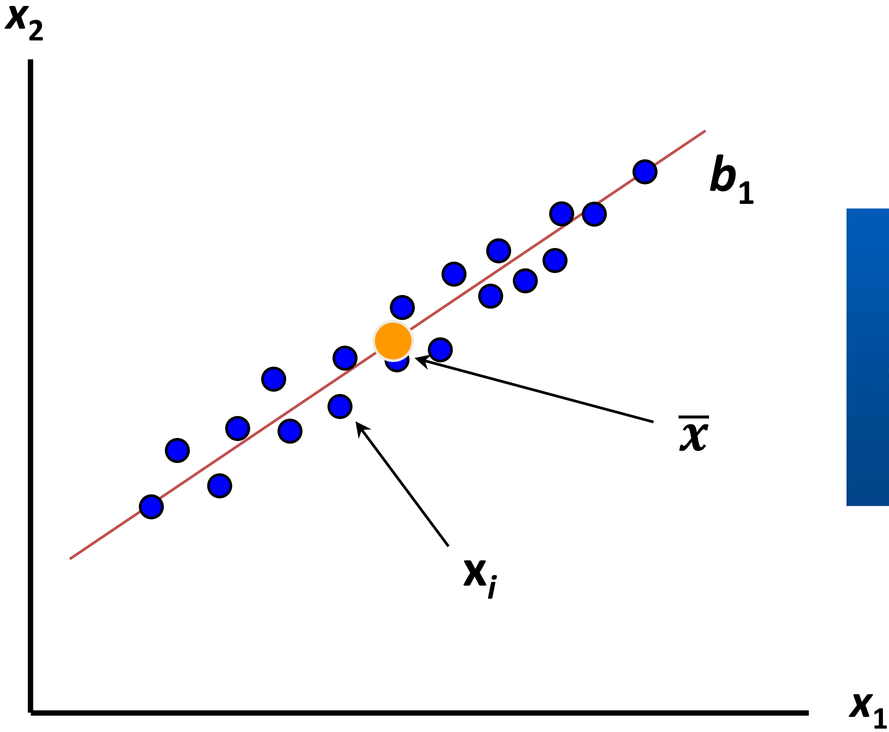


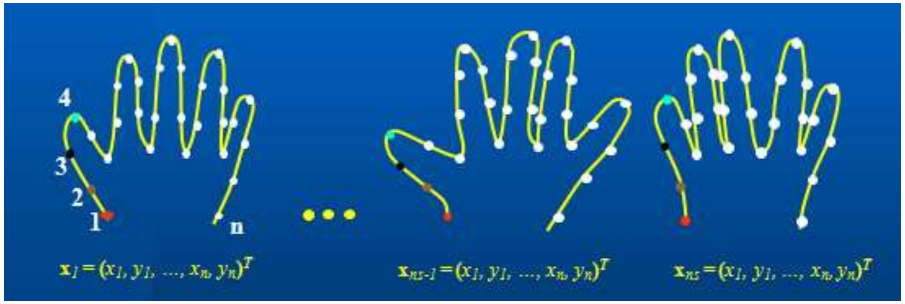
Example of shape configuration (left) and the configuration matrix (right) for a set of hand shapes.

Cootes, Taylor 1993

Modelling Shape Variability

Observation/Assumption: Points in shape population tend to move in **correlated** ways.





Cootes, Taylor 1993

Shape Alignment

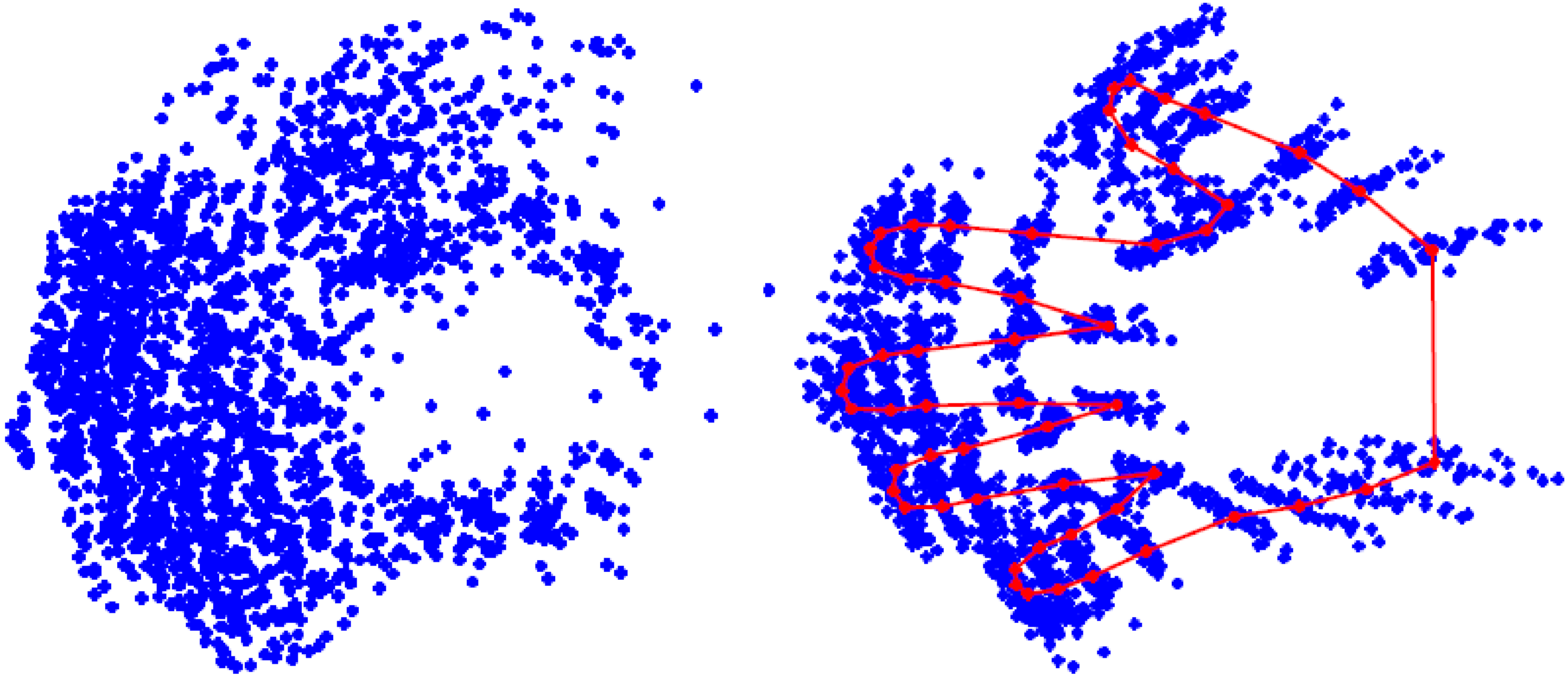
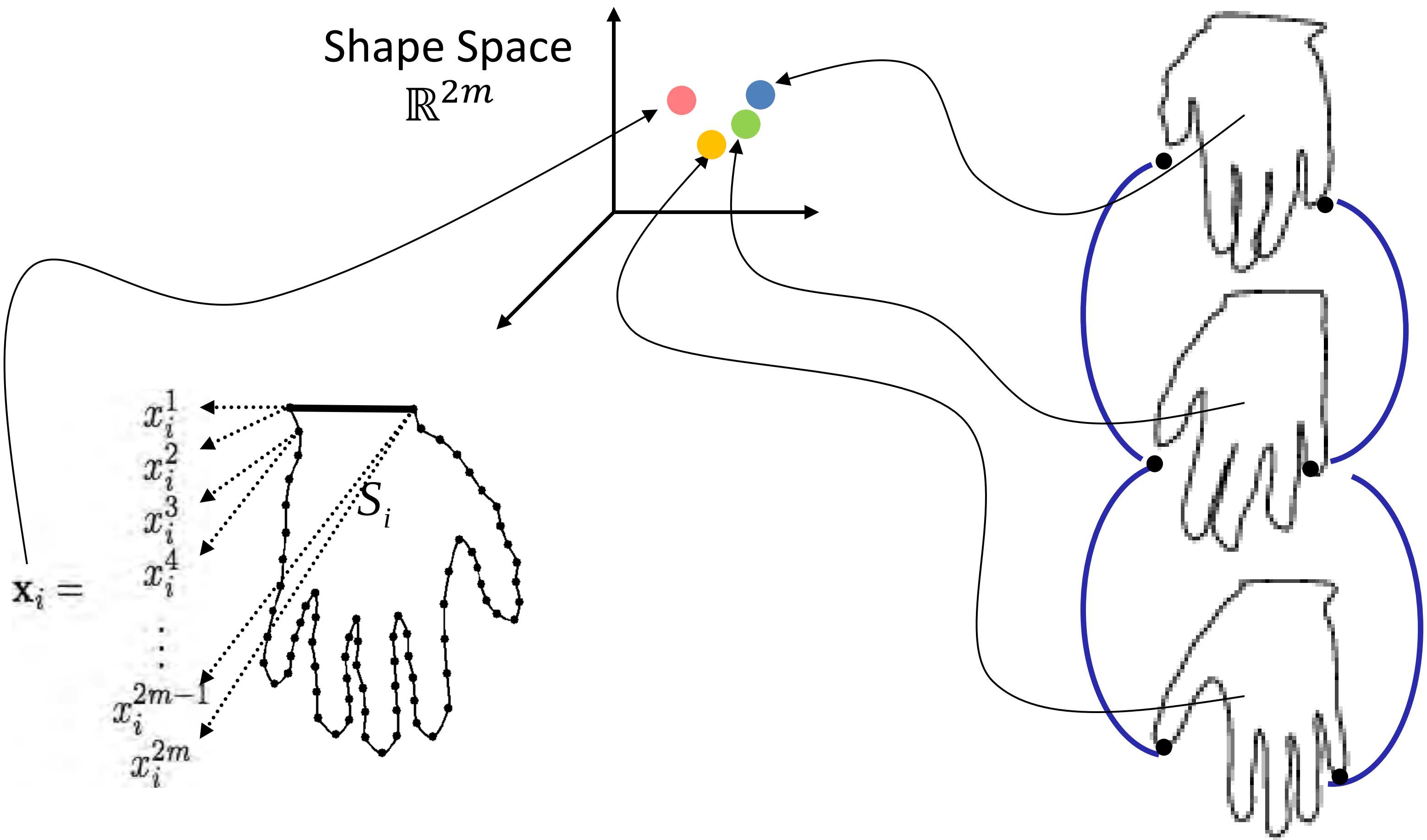


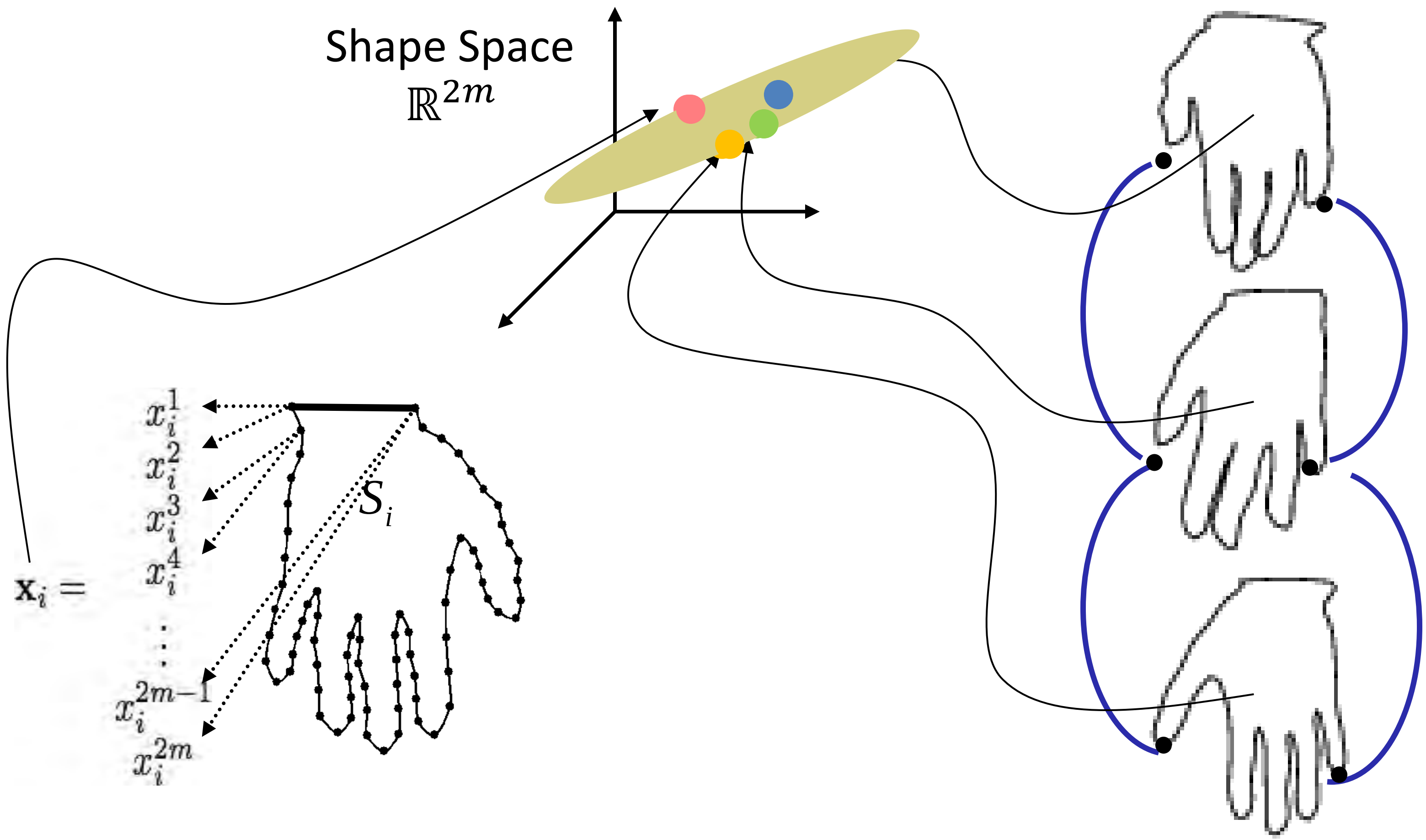
Figure 7: Left: 40 unaligned annotations. Right: 40 aligned annotations with mean shape in red.

Stegmann and Gomez, 2002

SSM and Shape Space



SSM and Shape Space: Correlation



Capturing the statistics of a set of aligned shapes

- Find mean shape $$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$
- Find deviations from the mean shape $$dx_i = \bar{x} - x_i$$
- Find covariance matrix $$S = \frac{1}{N}\sum_{i=1}^{N} dx_i dx_i^T$$
- Find eigenvalues/vectors of S $$Sp_k = \lambda_k p_k$$
- Modes of variation defined by eigenvectors $$p_k^T p_k = 1$$

Hand Model

Modes of shape variation

b_1 b_2 b

Courtesy of Chris Taylor, Cootes & Taylor 1992

Landmark Variability & Correlation Matrix

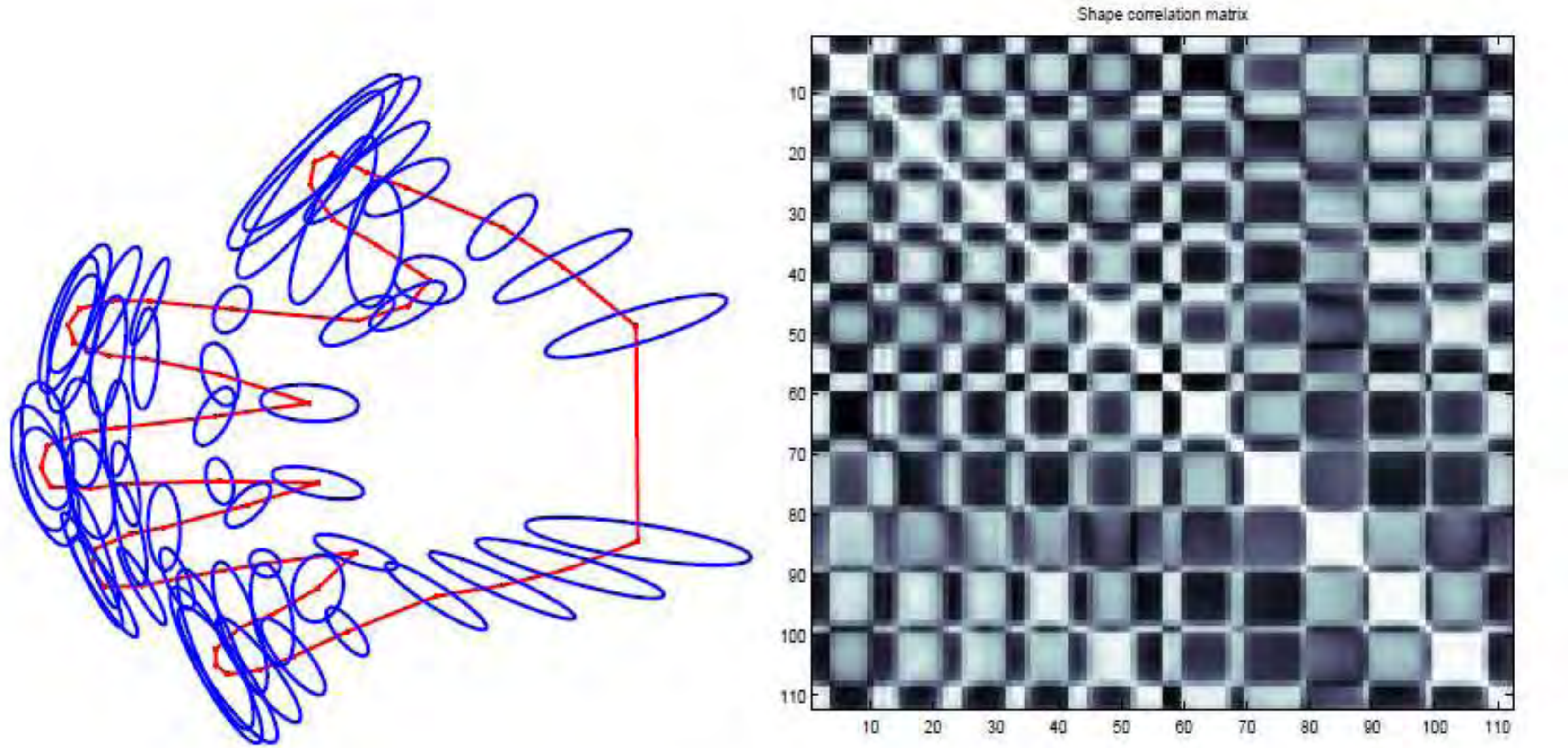


Figure 8: Left: Independent principal component analysis of each model point. Right: Correlation matrix of the annotations (white/grey/black maps to positive/none/negative correlation).

Stegmann and Gomez, 2002

Description in the Shape Space

- The modes of variation of the points of the shape are described by the eigenvectors of S:

$$x = \bar{x} + Pb$$

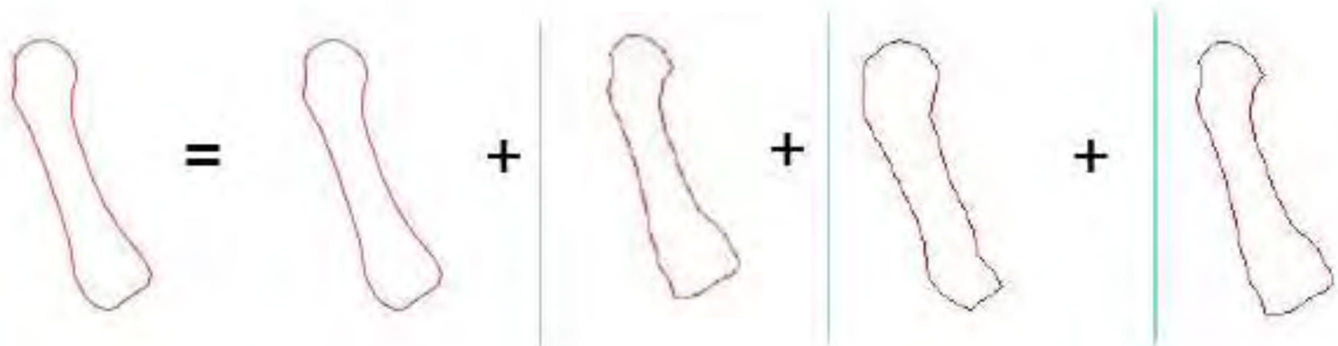
- Each shape is described by its weight vector b.

$$x - \bar{x} = Pb$$

$$b = P^T(x - \bar{x})$$

- The eigenvectors corresponding to the largest eigenvalue describe the most significant modes of variation in the training data.

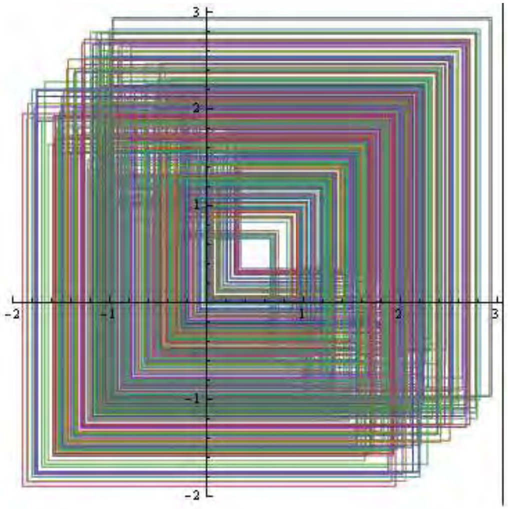
Shape Eigenbasis



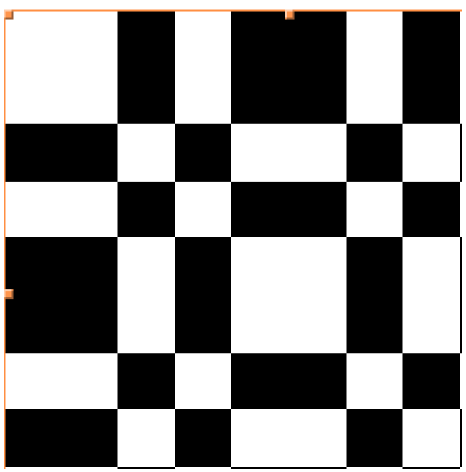
$$\mathbf{x}_{new} = \hat{\mathbf{m}} + b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$$

Slide credits: G. Langs

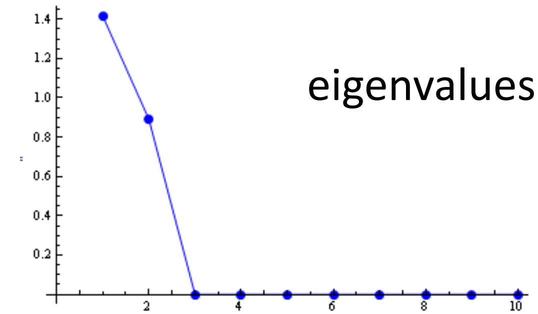
Synthetic Shapes: Translation

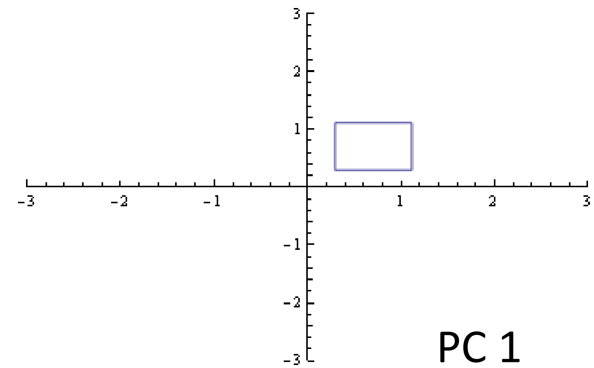


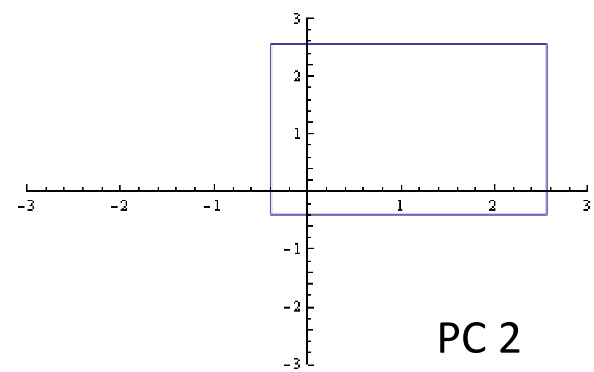
Input: Scaling/Translation



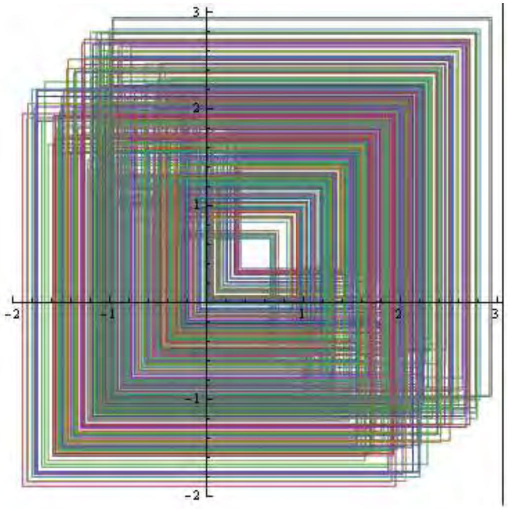
8*8 Correlation Matrix
[(x1,y1),(x2,y2),…,x4,y4)]



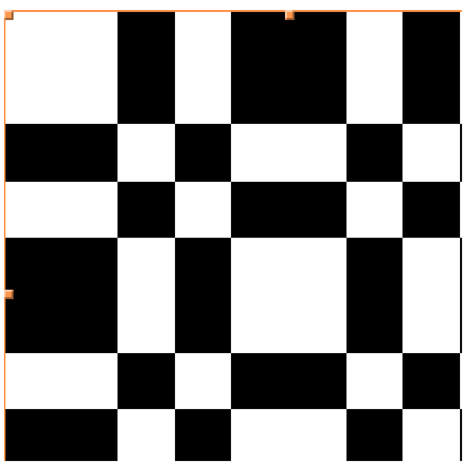




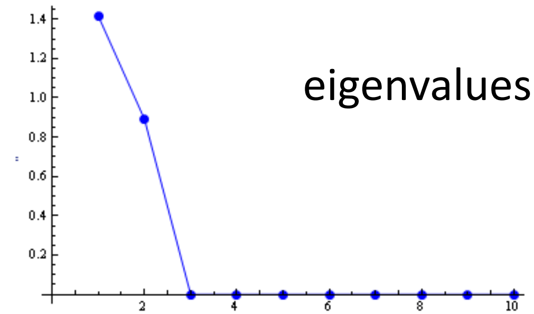
Synthetic Shapes: Translation

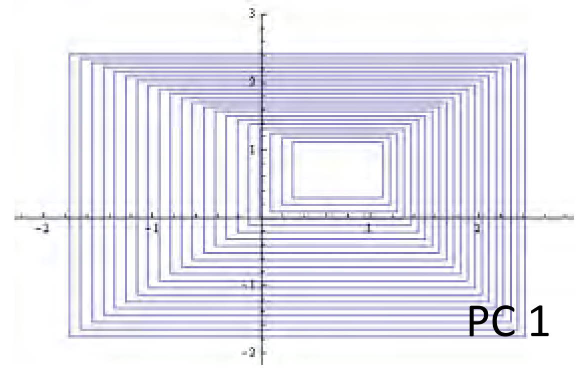


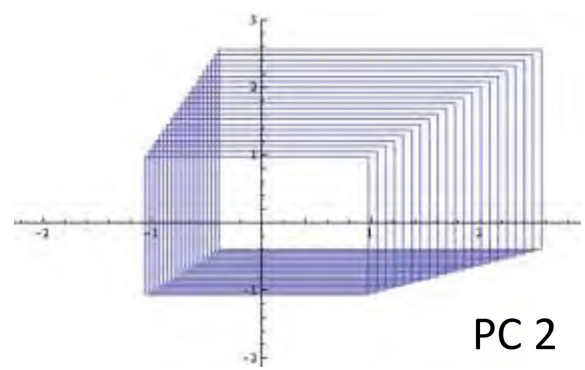
Input: Scaling/Translation



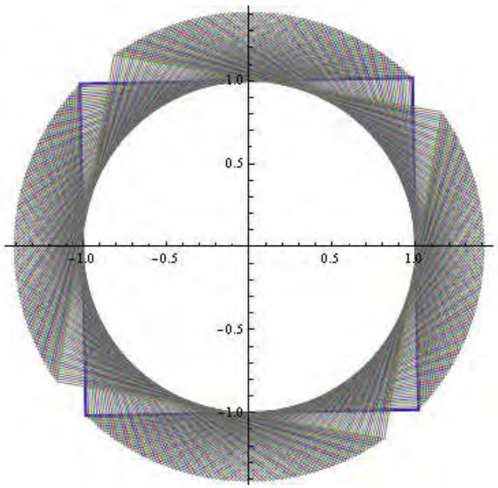
8*8 Correlation Matrix
[(x1,y1),(x2,y2),…,x4,y4)]



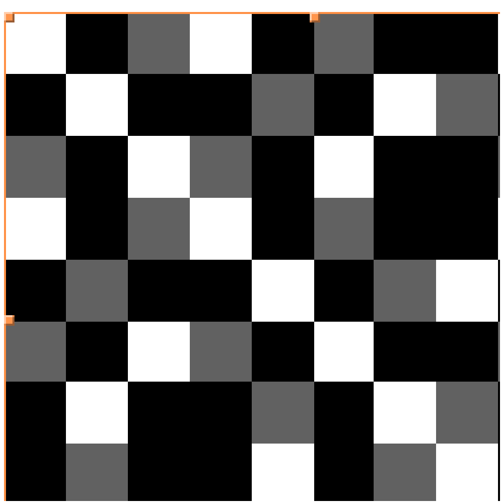




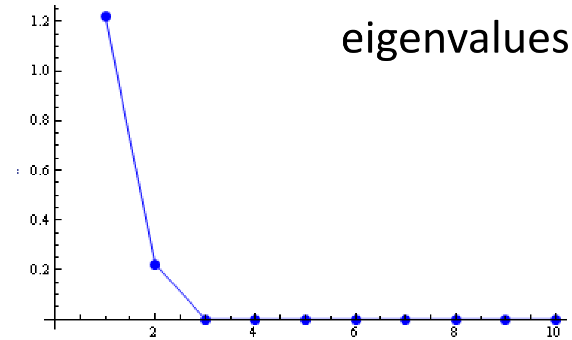
Synthetic Shape: Rotation?



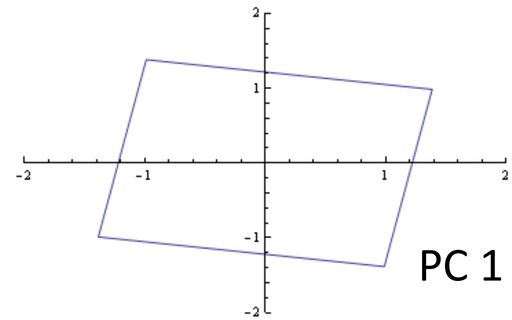
Input: Rotation of square by 80 deg.



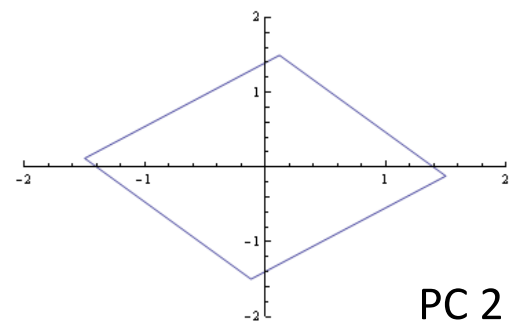
8*8 Correlation Matrix [(x1,y1),(x2,y2),…,x4,y4)].



eigenvalues

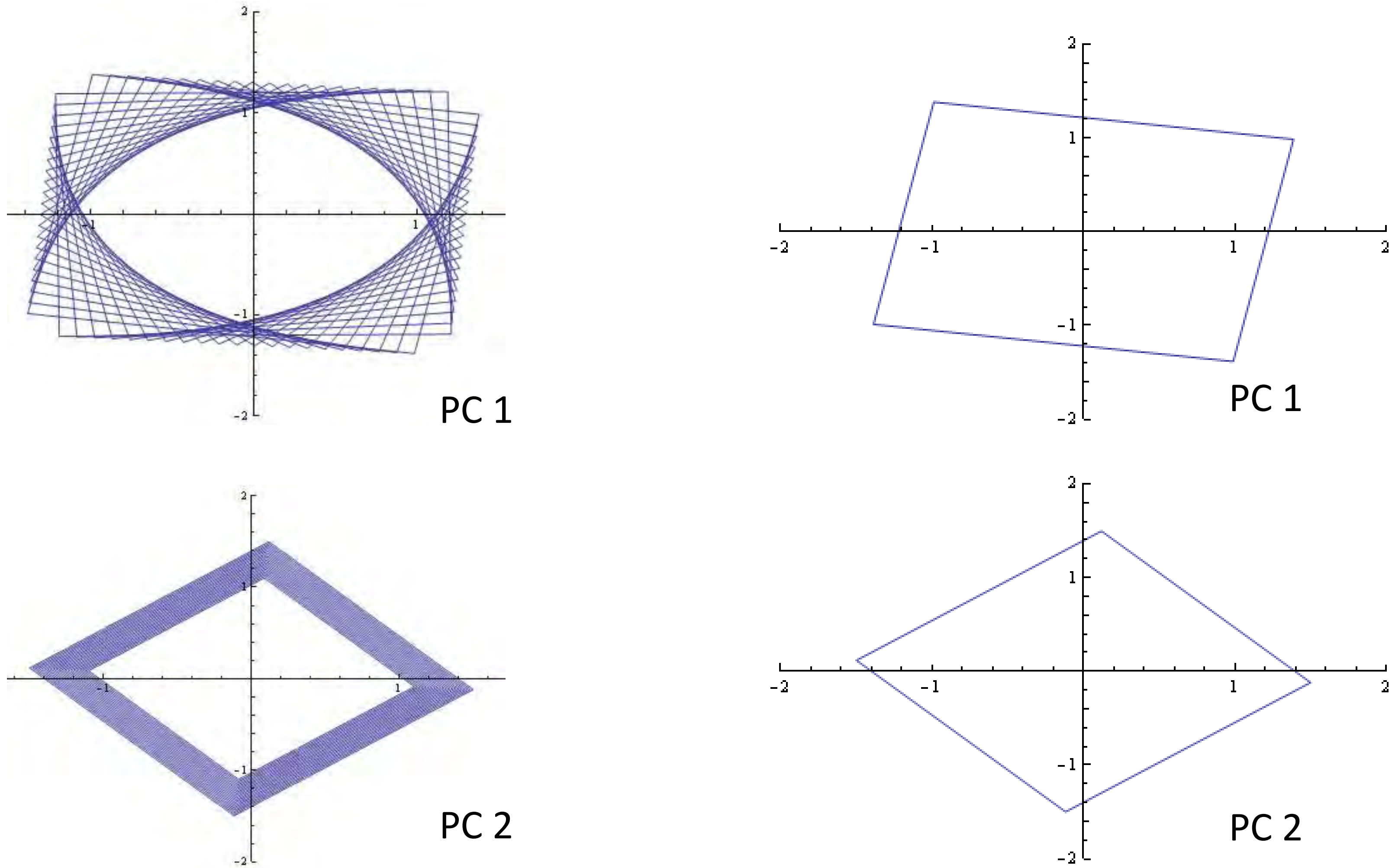


PC 1



PC 2

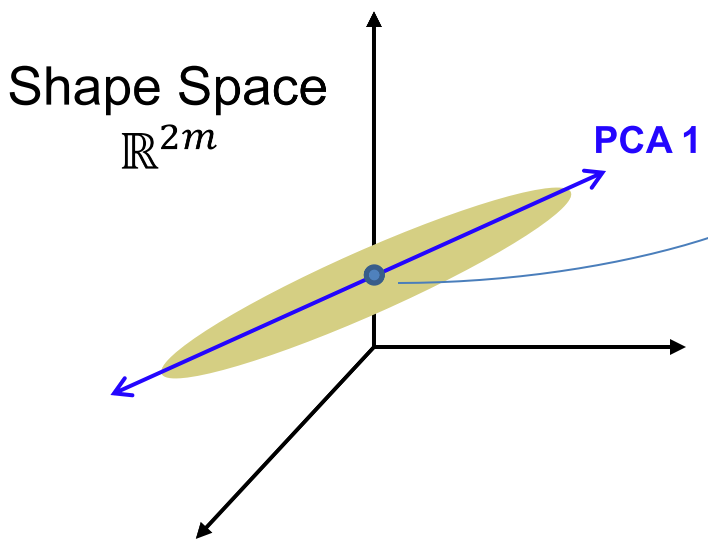
Synthetic Shape: Rotation (80deg)

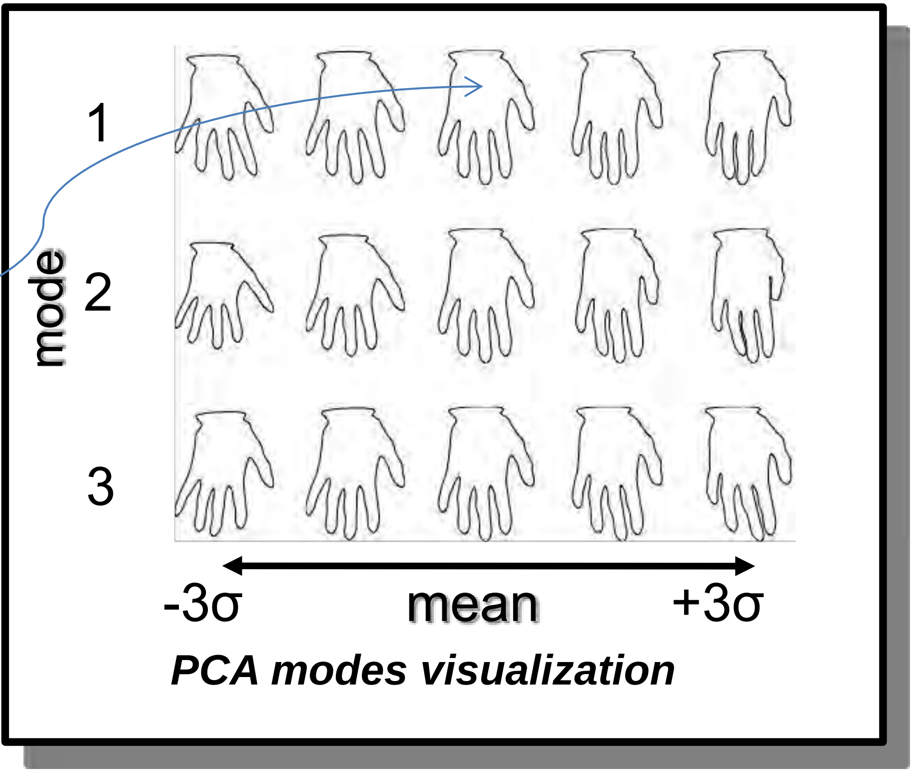


PCA in $\mathbb{R}^n$ generates linear subspaces V_k that maximize the variance of the projected data.

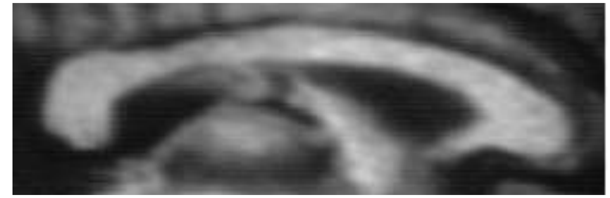
Statistics in Shape Space

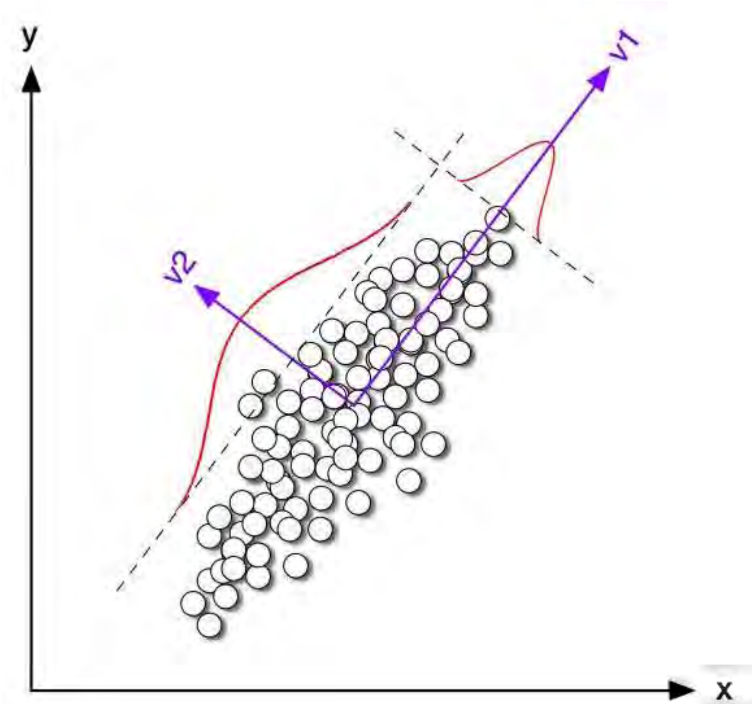
- Manual/automatic correspondences
- Gaussian models
- PCA for dimensionality in shape space

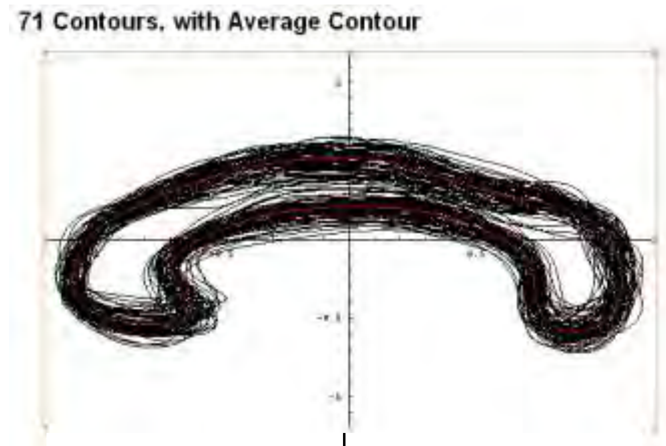


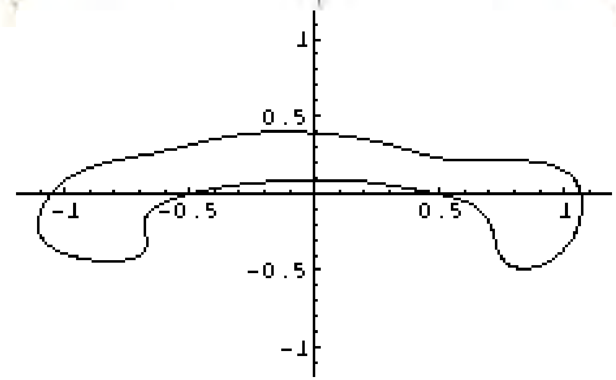


Summary Concept



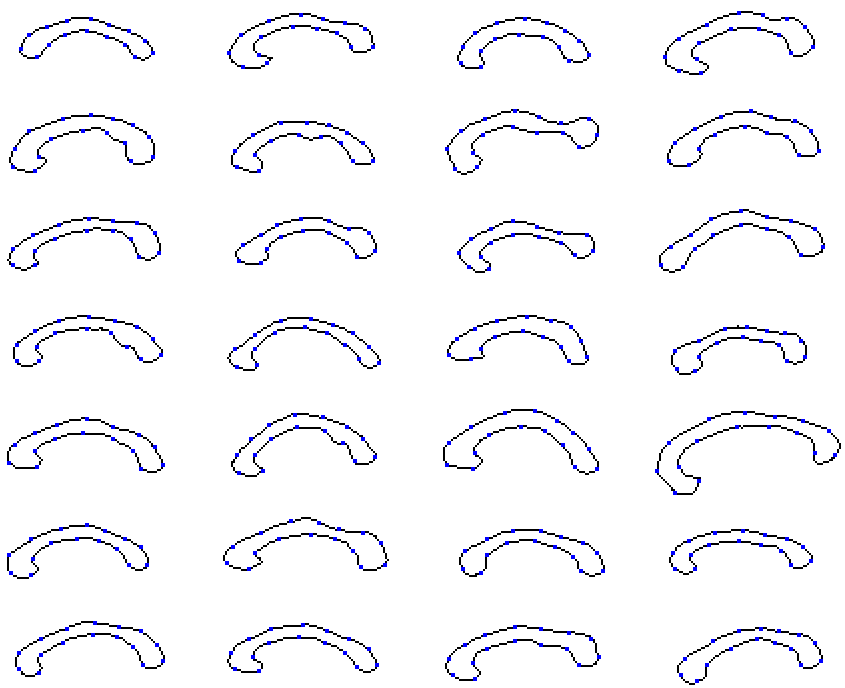


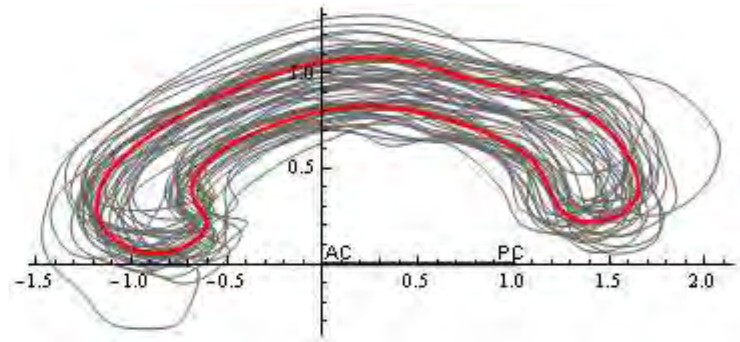


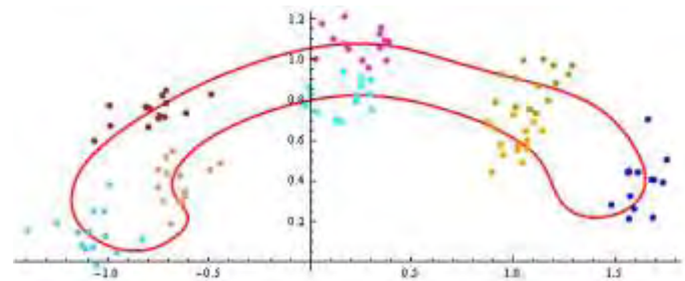


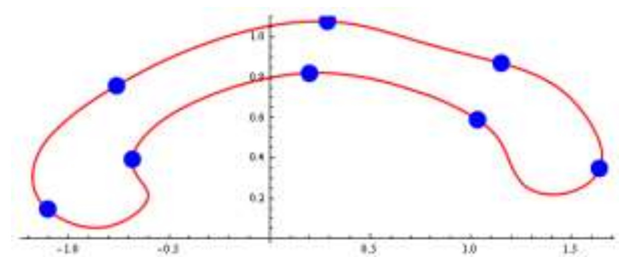
Compression/Feature selection: Project high dimensional measures into low-dimensional space of largest variability, few features → Statistics

Example: Corpus Callosum Study

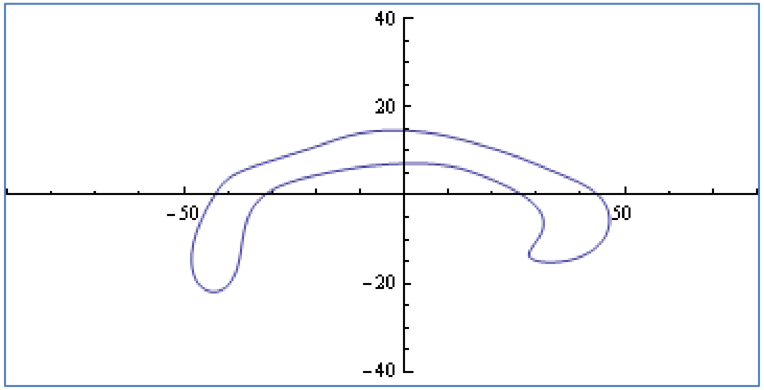




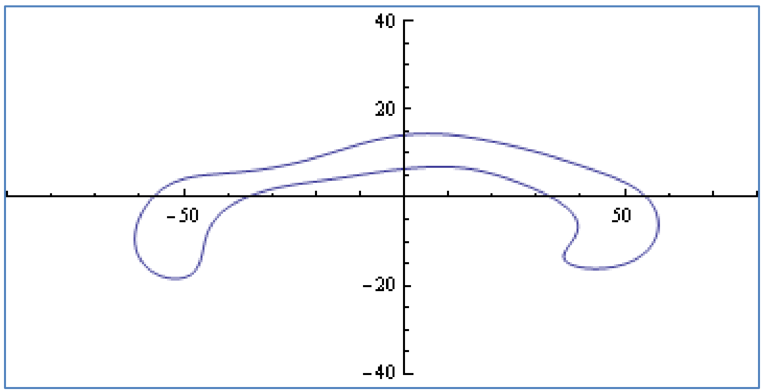




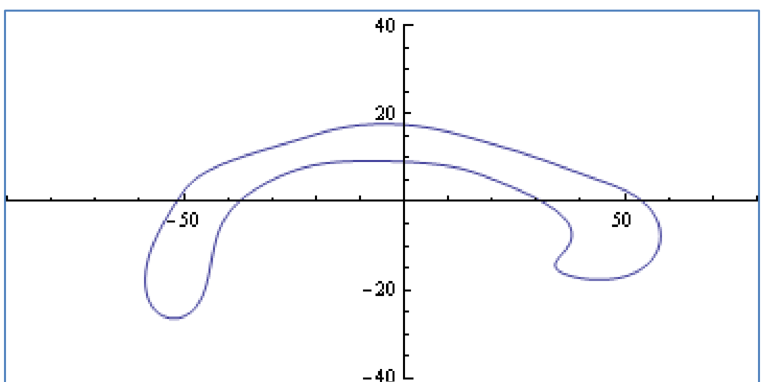
Boundary PCA



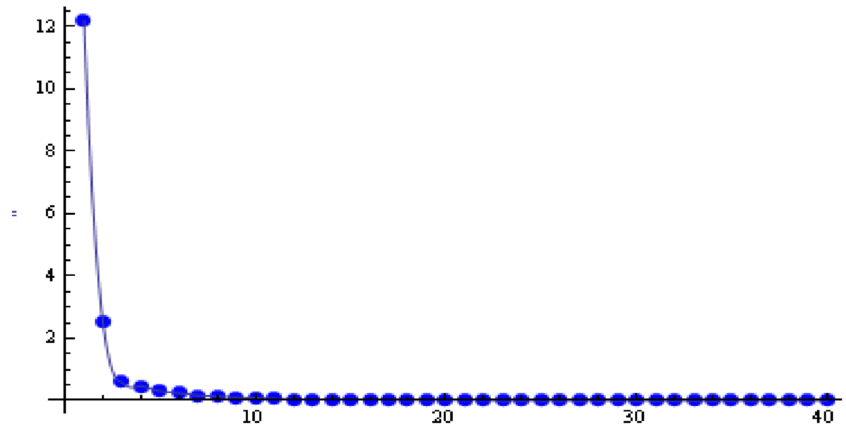
PCA 1



PCA 2

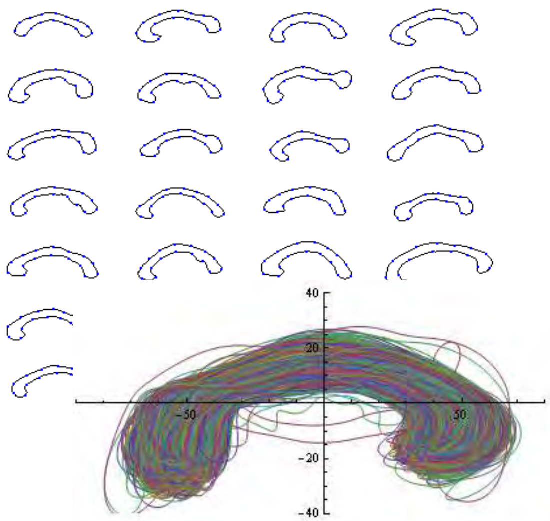


PCA 3

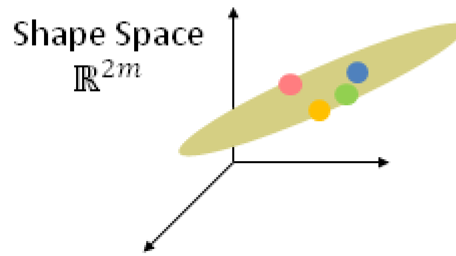


Eigenvalues:
95% of deformation energy is in the first 10 principal eigenmodes, and the first 2 represent 65% of the variation.

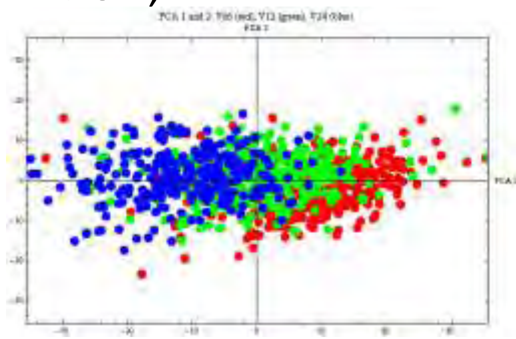
PCA Shape Space: Corpus Callosum Study

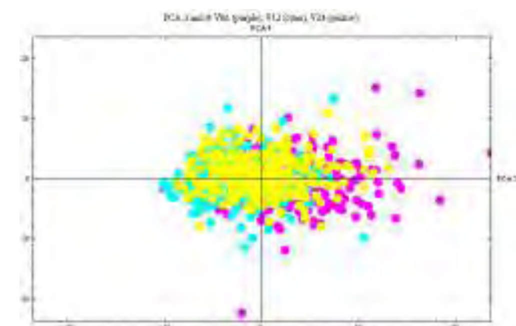


1040 infant CC shapes

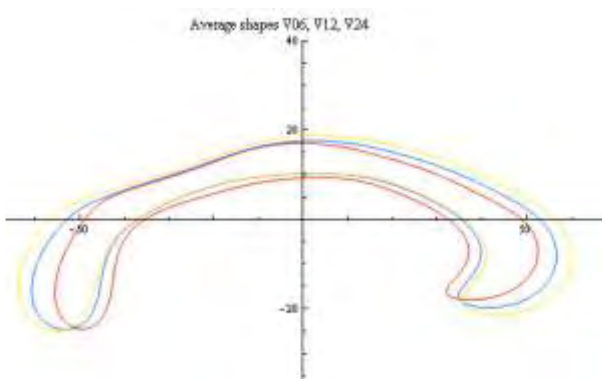


PC 1,2





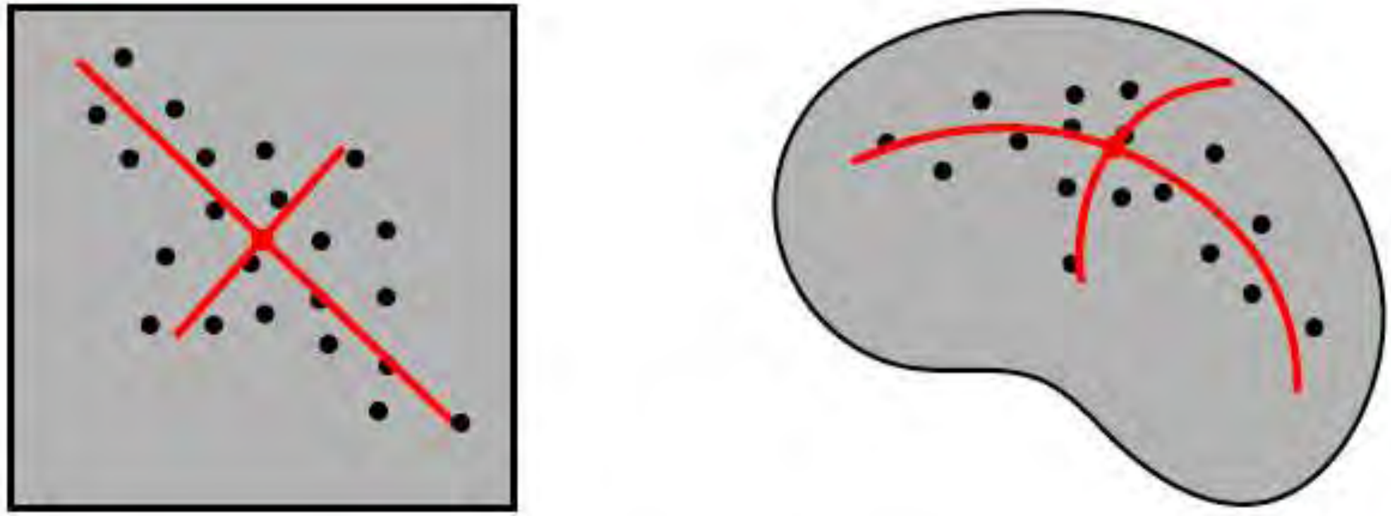
PC 3,4



Mean CC shapes: 6mo, 12mo, 24mo

ACE-IBIS Autism Study, UNC J. Piven

Generalizing PCA: Principal Geodesic Analysis



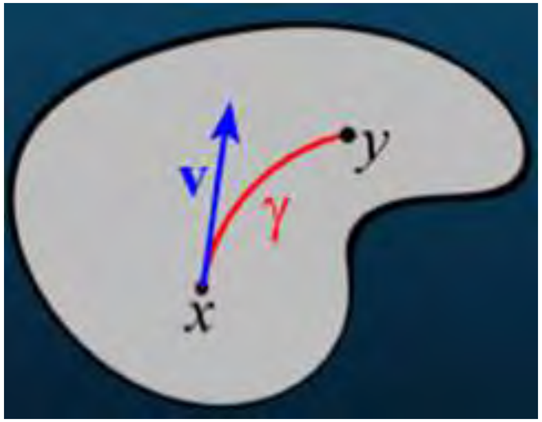
Linear Statistics
Principal Components Analysis
(PCA)

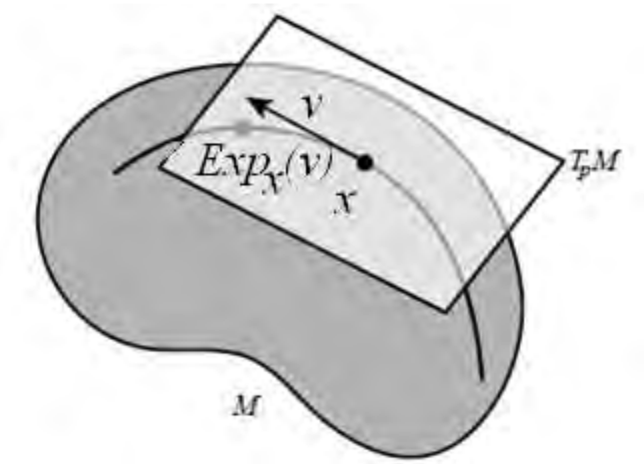
Curved Statistics
Principal Geodesics Analysis
(PGA)

Fletcher et al., IEEE TMI 2004

The Exponential Map

- We represent shapes as points on a manifold, rather than as points in Euclidean space.
- **Log map**: Function that computes a geodesic from two points on the manifold, representing the shortest path on the manifold between two points: $d(x, y) = Log_x(y) = \log(x^{-1}y)$.
- **Exponential map**: Function that computes points on the manifold from a base point and a vector in the tangent space: $Exp_x(v) = xexp(y)$.





Intrinsic Means (Fréchet)

The intrinsic mean of a collection of points $x_1 \cdots x_N$ in a metric space M is

$$\mu = \arg\min_{x \in M} \sum_{i=1}^{N} d(x, x_i)^2,$$

where $d(.,.)$ denotes distance in M.

PGA

- PGA is the natural generalization of PCA to a manifold space.
- Covariance matrix is constructed with the tangent vectors at the Fréchet mean (vectors v_i).
- Fréchet mean: No closed form solution in this space, iterative procedure:

choose an initial guess for μ

for k=1 to number of iterations

$$v_i = \mathrm{Log}_{\mu_k}(x_i)$$

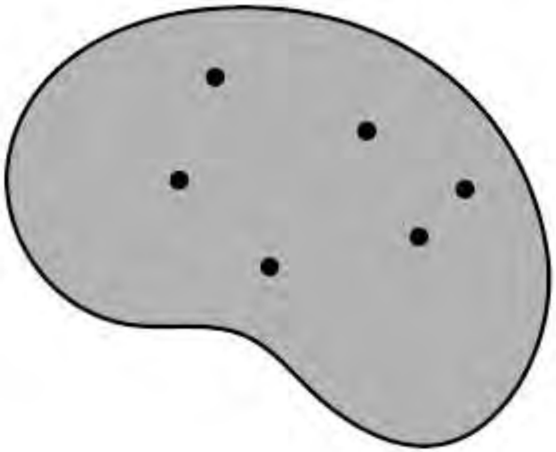
$$\hat{v} = \frac{1}{N}\sum_i v_i$$

$$\mu_{k+1} = \mathrm{Exp}_{u_k}(\hat{v})$$

end

Algorithm for computing Frechet mean on the manifold.

Computing Means



Gradient Descent Algorithm:

Input: $\mathbf{x}_1, \ldots, \mathbf{x}_N \in M$

$\mu_0 = \mathbf{x}_1$

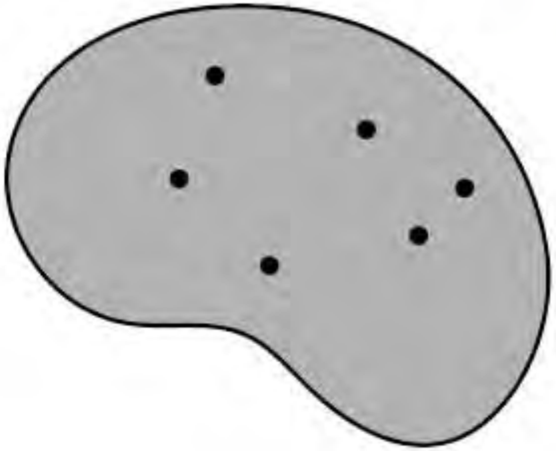
Repeat:

$$\Delta\mu = \frac{1}{N} \sum_{i=1}^{N} \mathrm{Log}_{\mu_k}(\mathbf{x}_i)$$

$$\mu_{k+1} = \mathrm{Exp}_{\mu_k}(\Delta\mu)$$

Fletcher et al., IEEE TMI 2004

Computing Means



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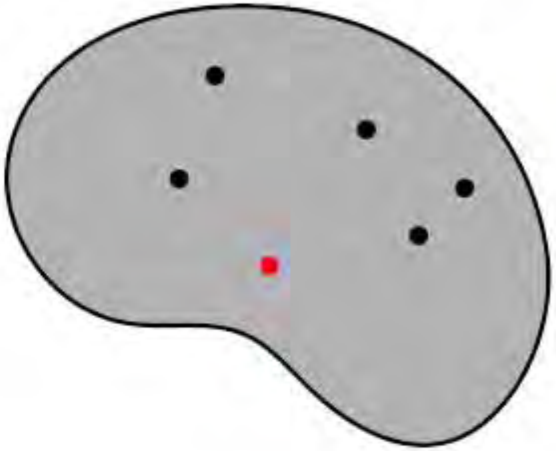
Fletcher et al., IEEE TMI 2004

Computing Means



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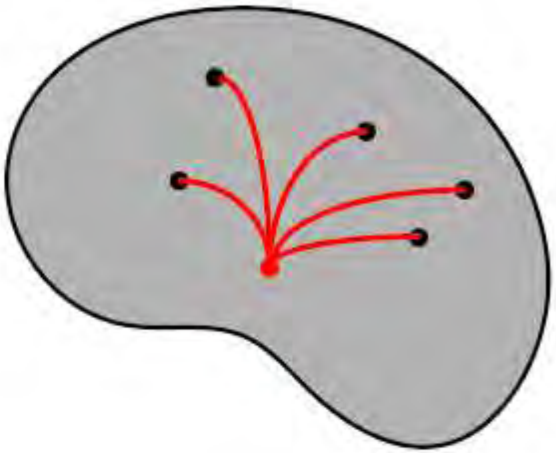
Fletcher et al., IEEE TMI 2004

Computing Means



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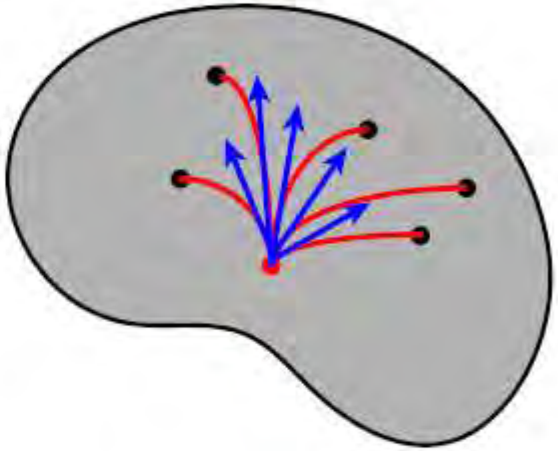
Fletcher et al., IEEE TMI 2004

Computing Means



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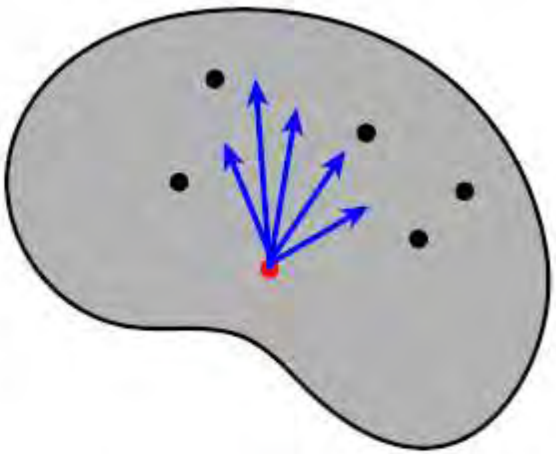
Fletcher et al., IEEE TMI 2004

Computing Means



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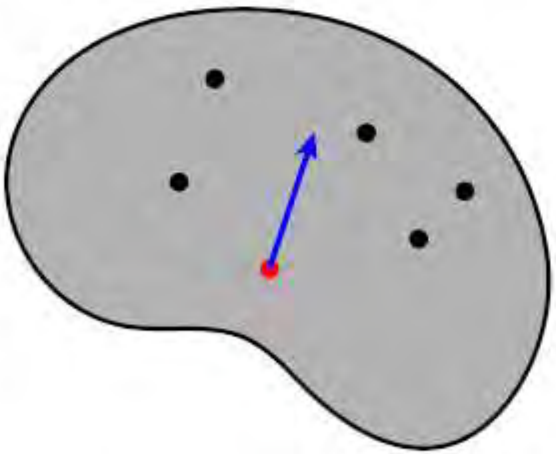
Fletcher et al., IEEE TMI 2004

Computing Means



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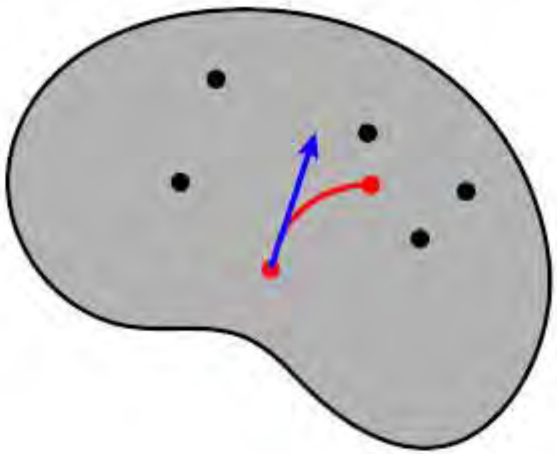
Fletcher et al., IEEE TMI 2004

Computing Means



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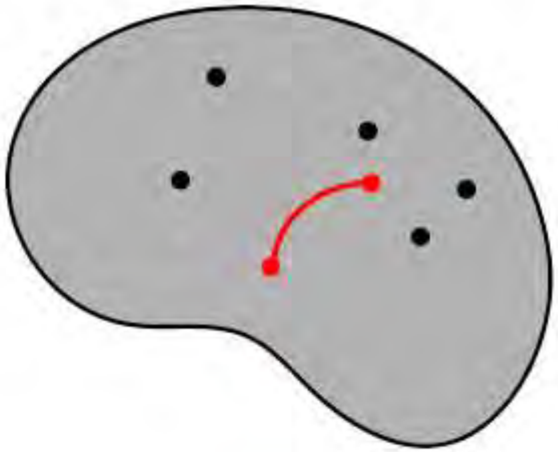
Fletcher et al., IEEE TMI 2004

Computing Means



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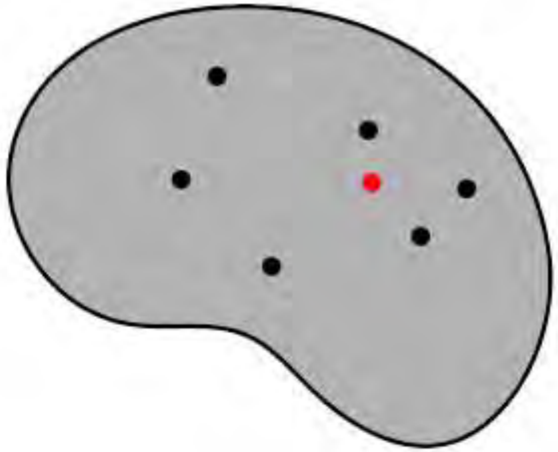
Fletcher et al., IEEE TMI 2004

Computing Means



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Repeat:

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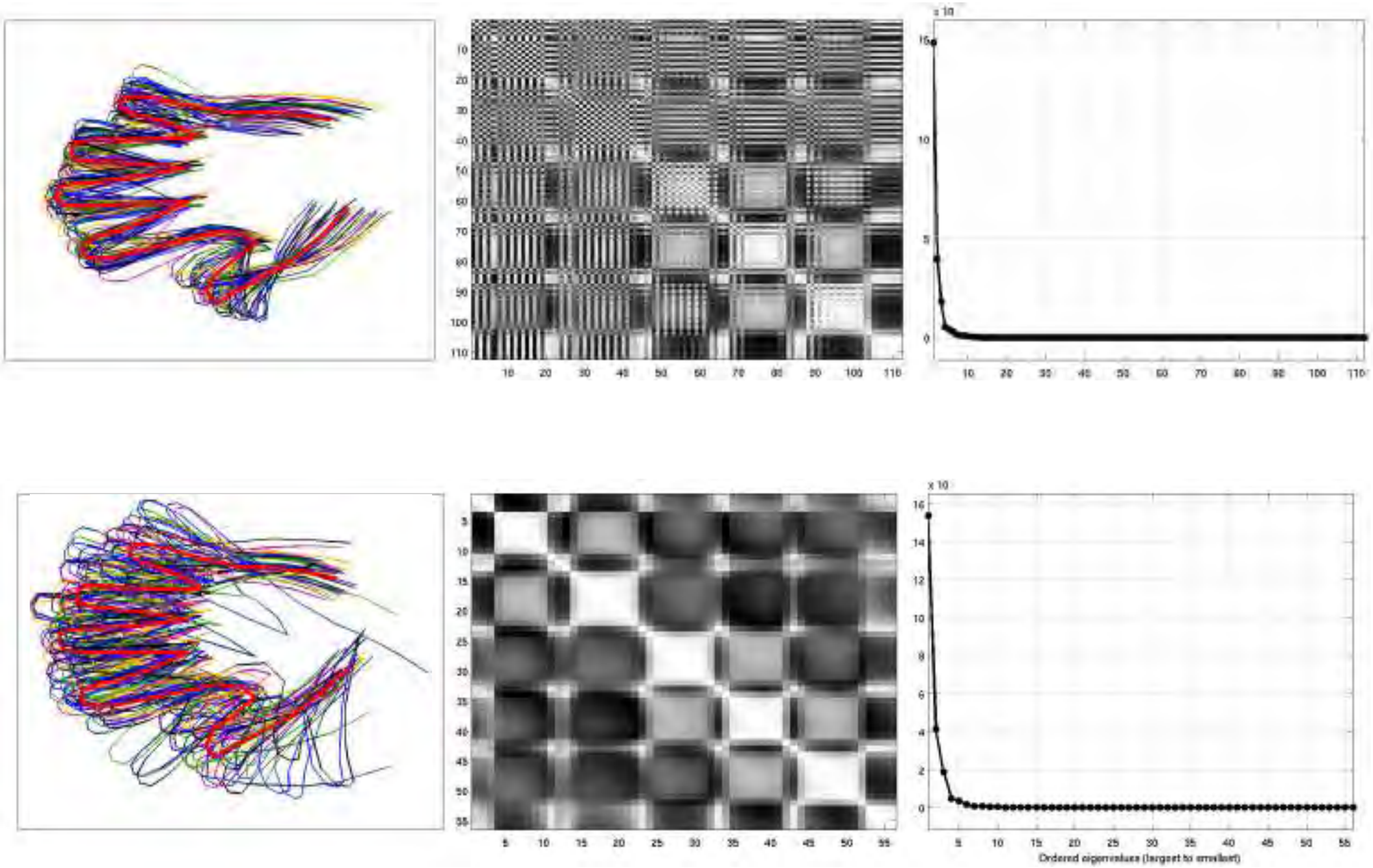
$$\mu_{k+1} = \mathrm{Exp}_{\mu_k}(\Delta\mu)$$

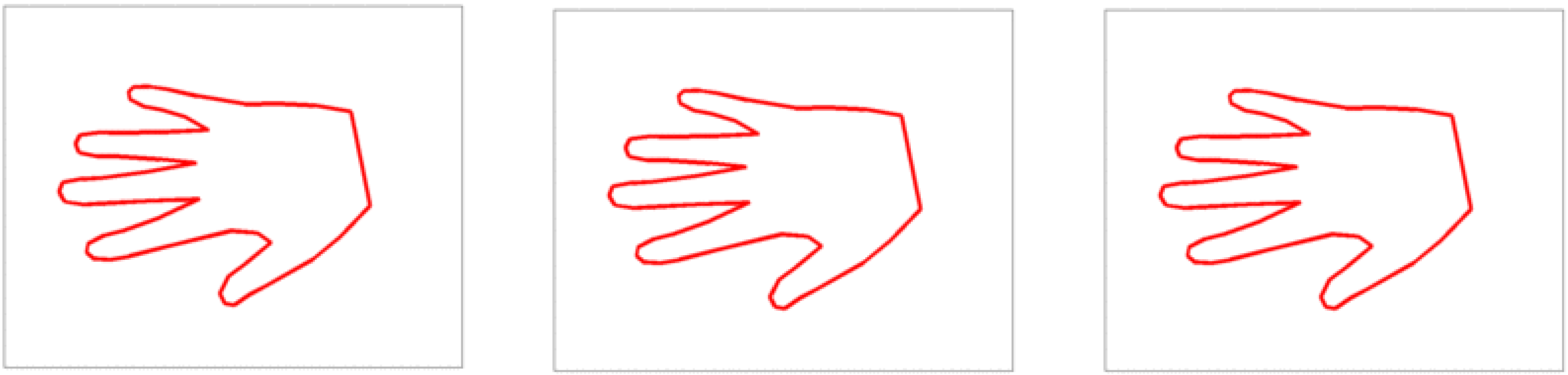
Fletcher et al., IEEE TMI 2004

Comparison PCA-PGA

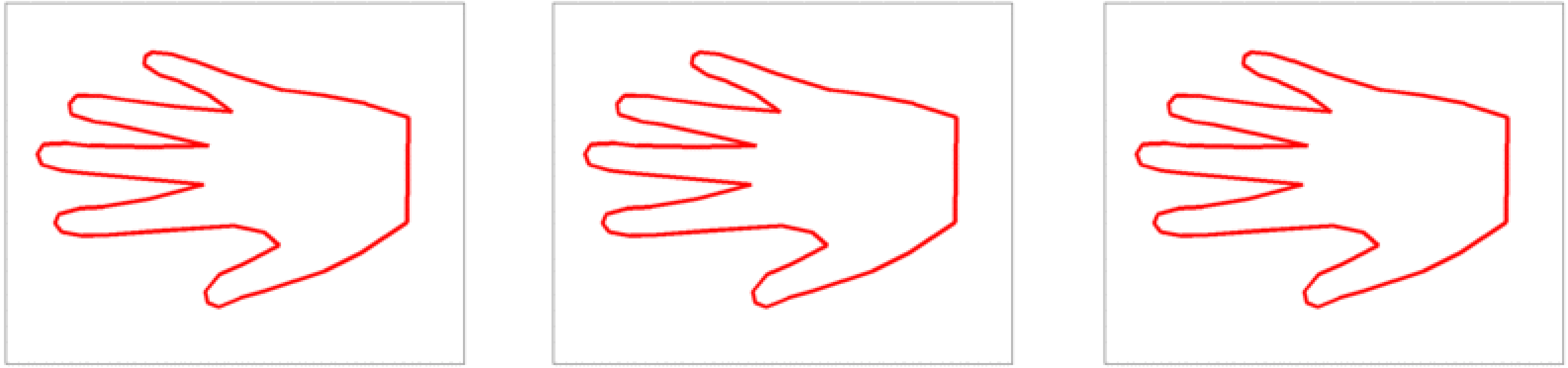


Comparison PCA-PGA

PCA with Procrustes



PGA



Discussion:

- Qualitatively slightly different but no obvious major differences.
- Details are in the math: PGA guarantees by definition rigid invariance (rotation, scale), PCA after Procrustes shows slight amount of scale differences but none for rotation.

Non Unimodal Shape Space: Gaussian Mixture Model

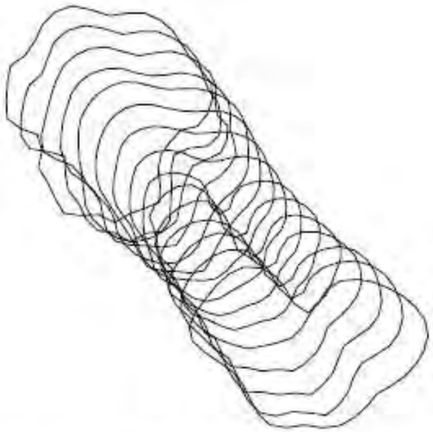


Figure 9: Contours from sequential slices

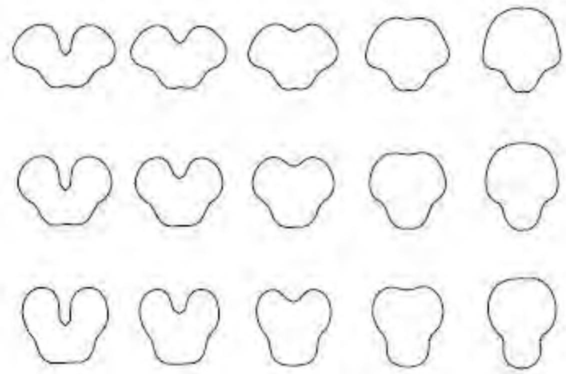


Figure 10: Shape for b_1 vs b_2 for brain stem

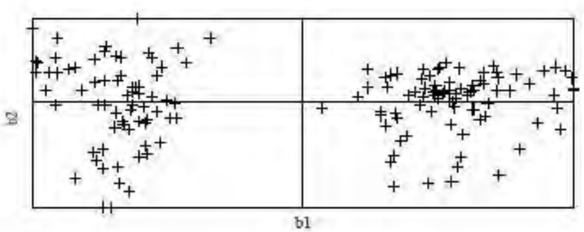


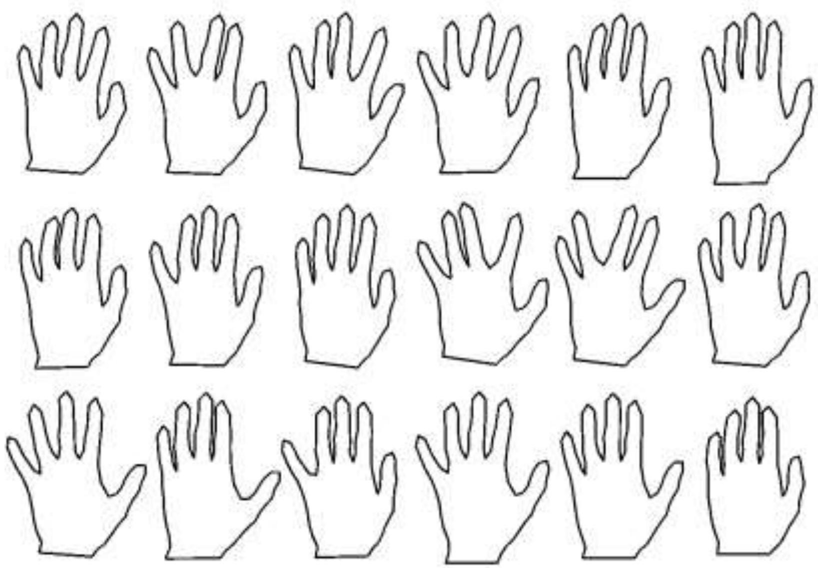
Figure 11: Plot of b_1 vs b_2 for brain stem

Figure 12: pdf approximation with 2 gaussians

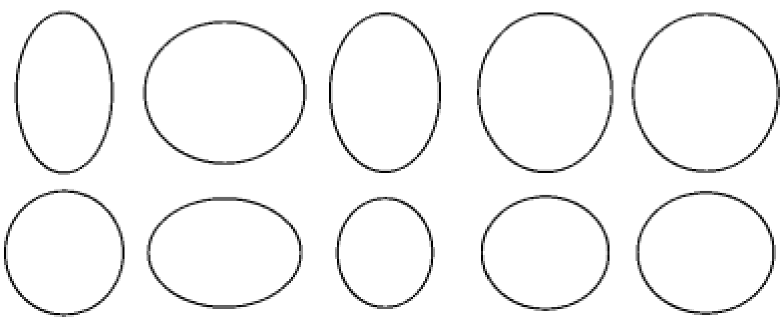
A Mixture Model for Representing Shape Variation, Cootes et al., IVC 1999

Towards Robust Statistics on Shapes

Example: Complex Projective Kendal Shape Space

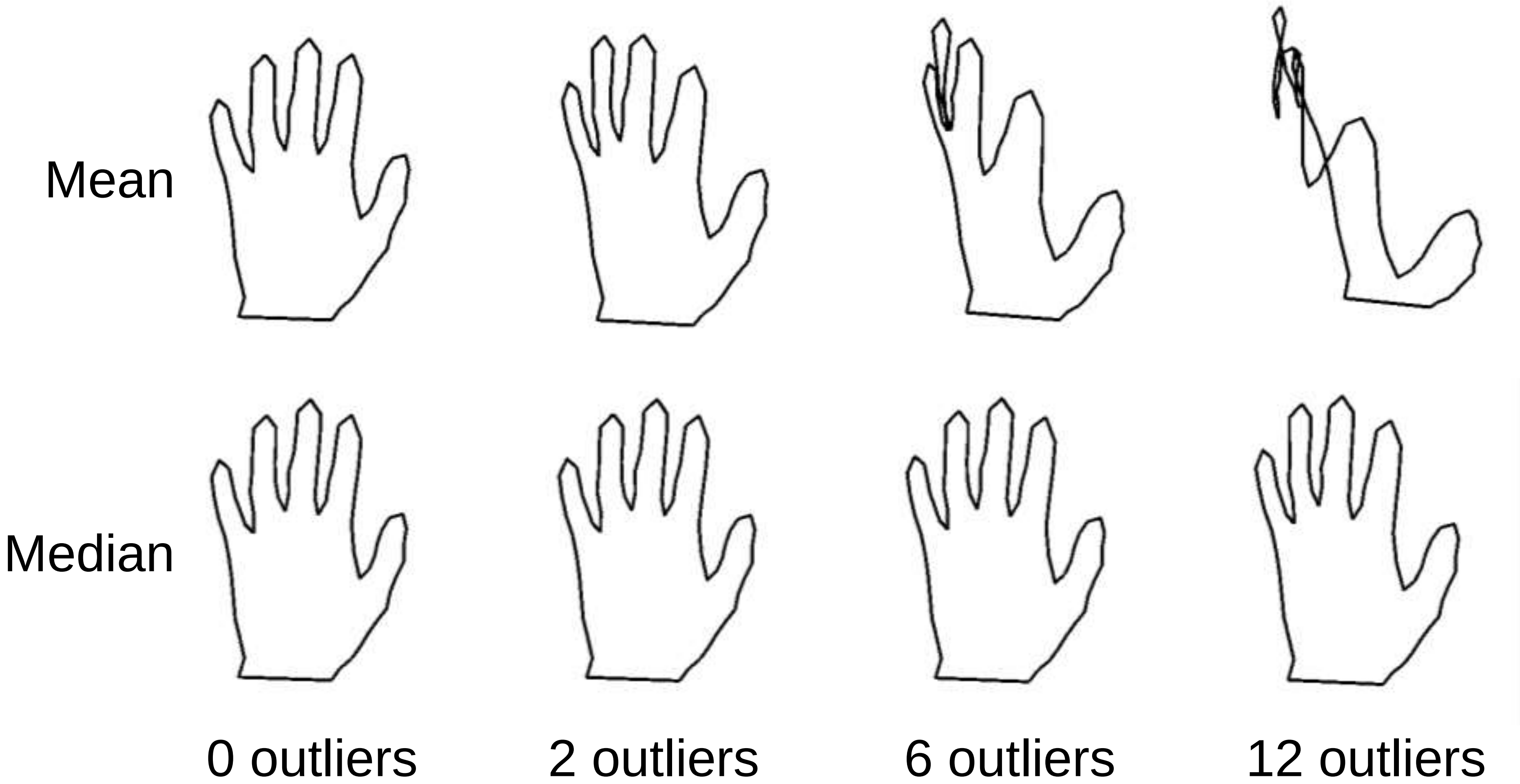


Input Data: 18 Hand Outlines (Cootes & Taylor)



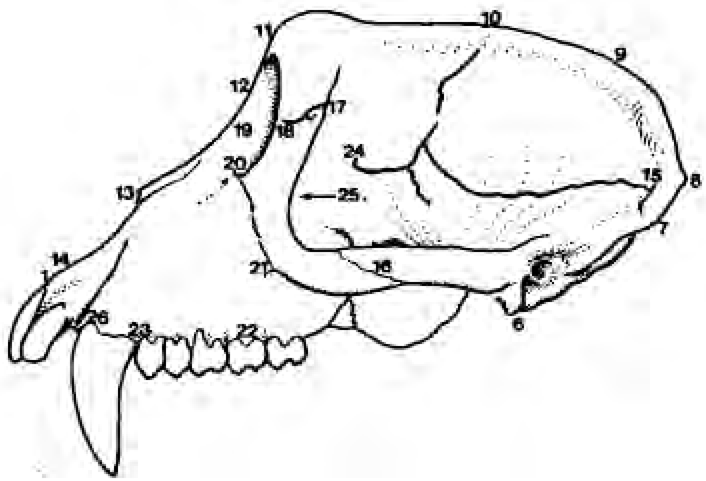
Outliers: random ellipses

Towards Robust Statistics on Shapes



Sarang Joshi, Utah

Landmarks / Homologous Points



From Dryden & Mardia

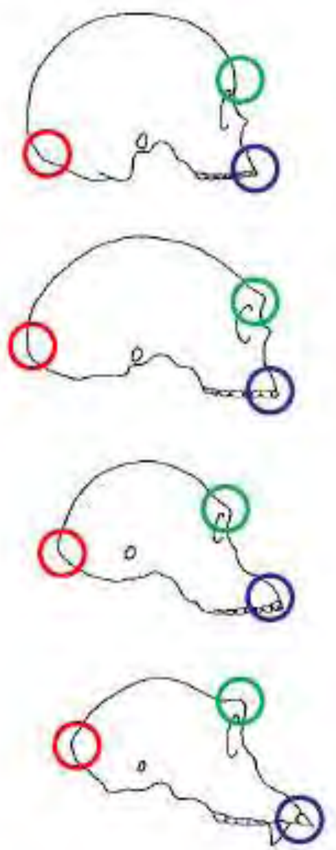
- A **landmark** is an identifiable point on an object that corresponds to matching points on similar objects.
- This may be chosen based on the application (e.g., by anatomy) or mathematically (e.g., by curvature).

Landmarks ctd.

- **Anatomical landmarks** are points assigned by an expert that corresponds between objects of study in a way meaningful in the context of the disciplinary context.
- **Mathematical landmarks** are points located on an object according some mathematical or geometrical property, i.e. high curvature or an extremum point.
- **Pseudo-landmarks** Constructed points on an object either on the outline or between landmarks.

[Dryden & Mardia]

Landmark Correspondence

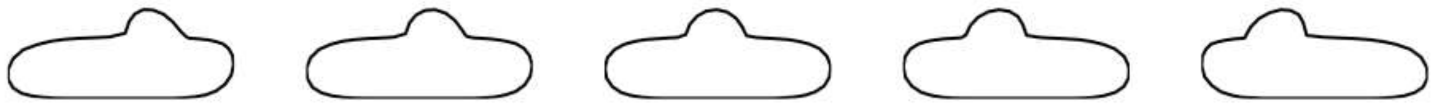


From C. Small, *The Statistical Theory*

Homology:

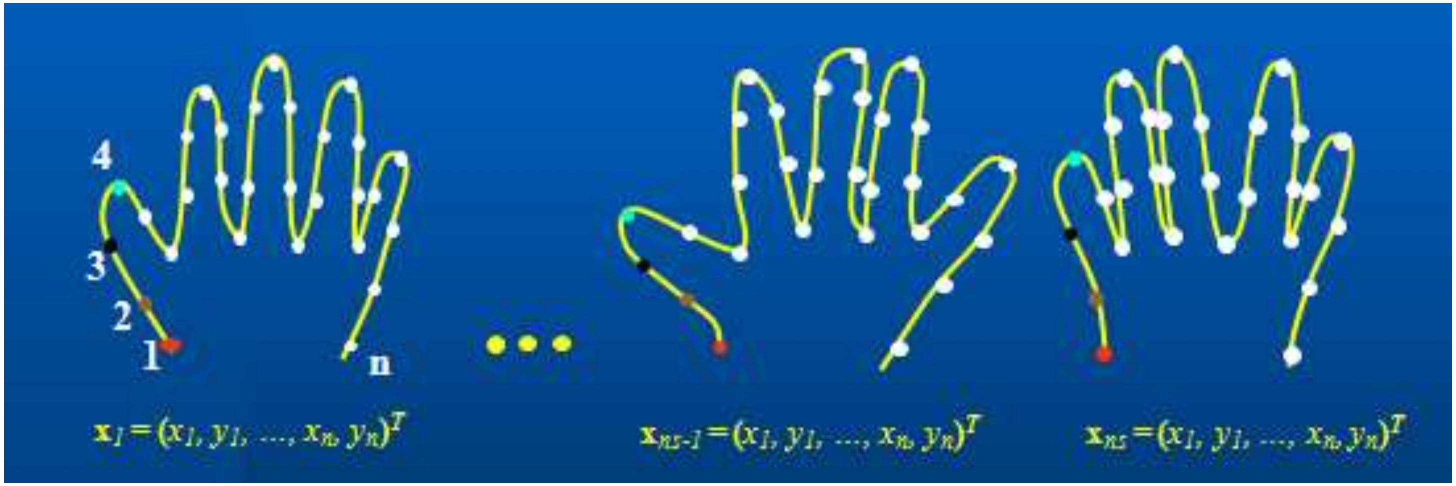
Corresponding (homologous) features on skull images.

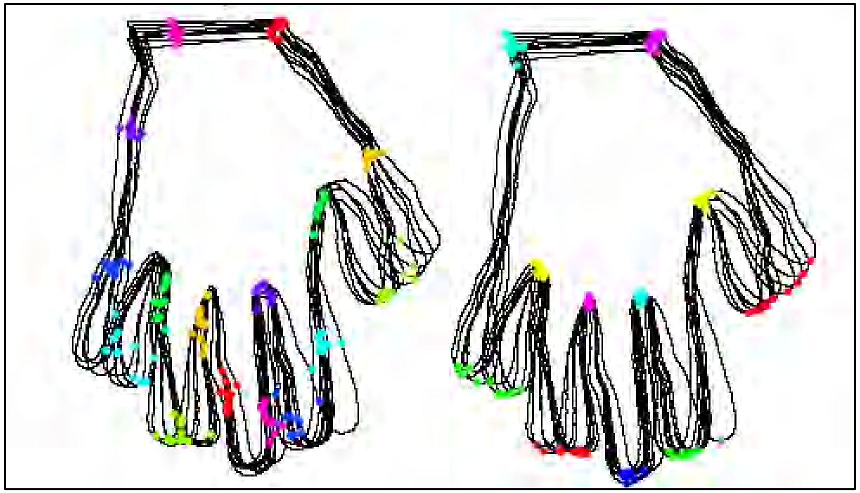
Correspondences and Shape



- The choice matters
  - Defines the shape space
- Manual landmarks
  - Not practical
  - 3D, not clear
  - User error
- Need: automatic 2D/3D correspondence placement
  - Computational concept?

“Good” and “Bad” Correspondence





“Good” placement:

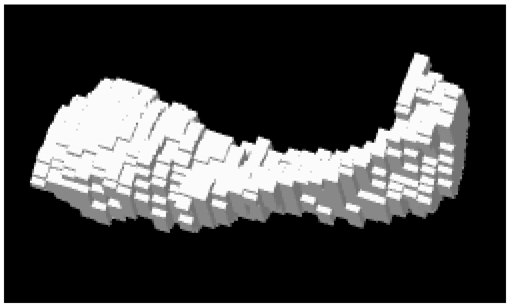
- Reduced variability.
- May lead to better, more compact statistical shape models.

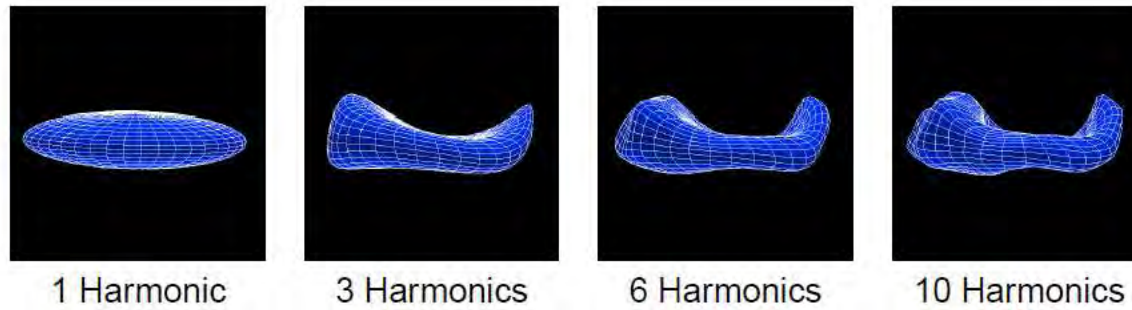
Left: Arc-length parametrization

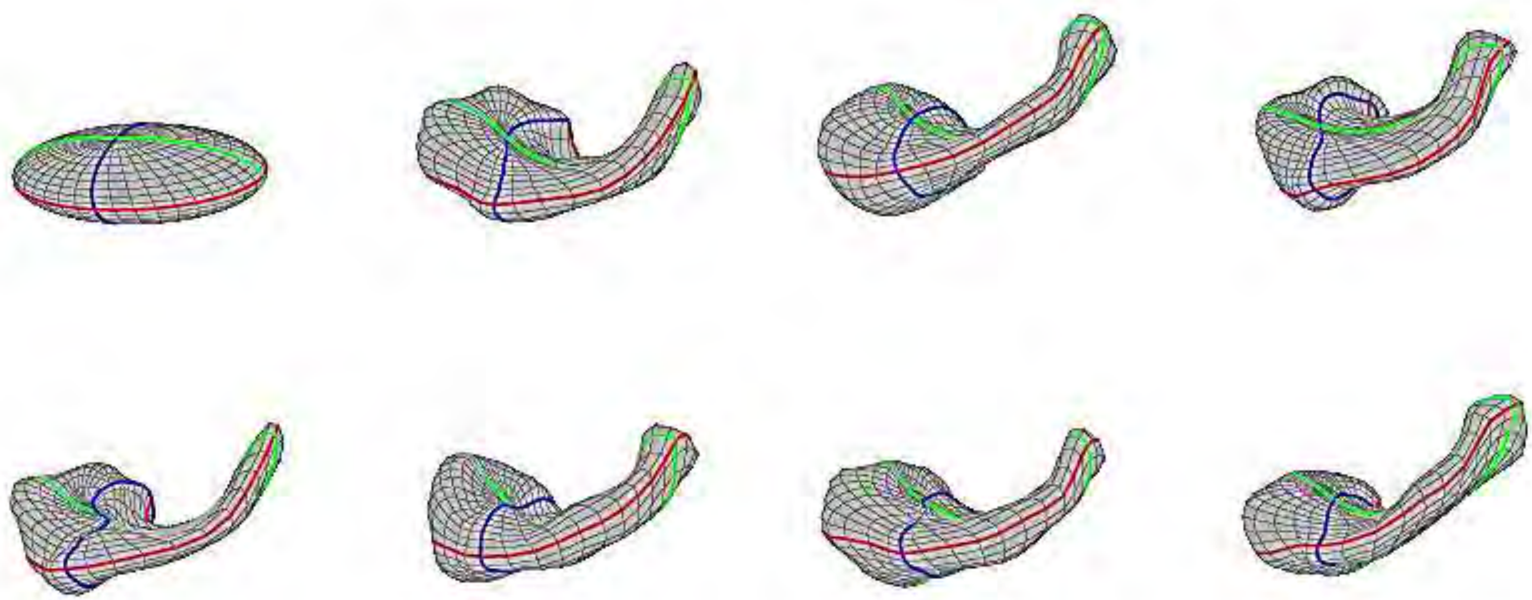
Right: Manual placement of corresponding landmarks

From: PhD thesis Rhodri Davies

Spherical Harmonics: Correspondence via Parametrization



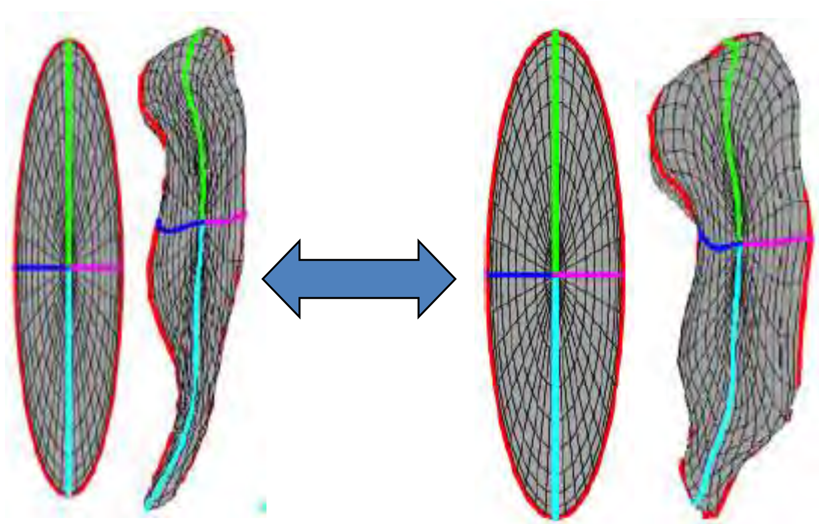


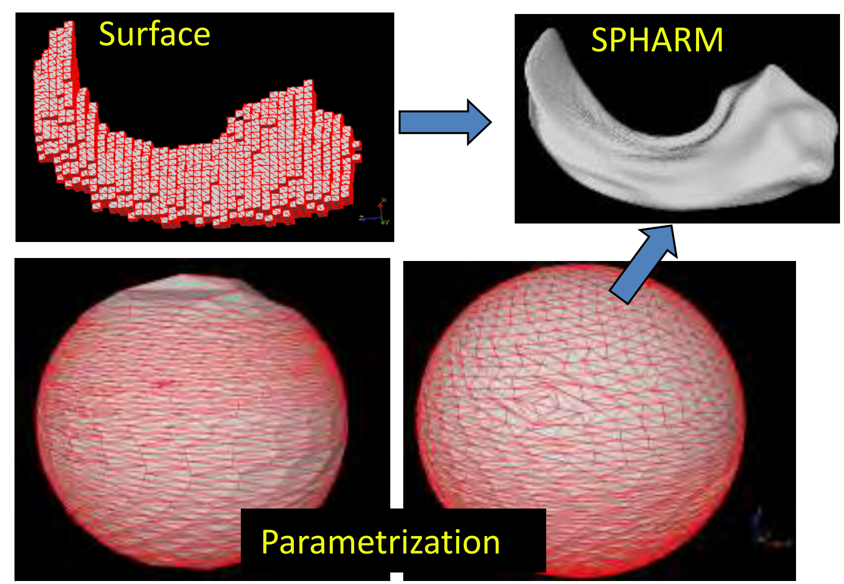


Szekely, Kelemen, Brechbuehler, Gerig, MedIA 1996

Correspondence: SPHARM

- Correspondence by same parameterization
  - Area ratio preserving through optimization
  - Location of meridian and equator ill-defined
- Poles and Axis of first order ellipsoid
- Object specific, independent, but sensitive to objects with rotational symmetry/ambiguity





Correspondence and quality of shape model

Manual placement | Arc-length parametrization

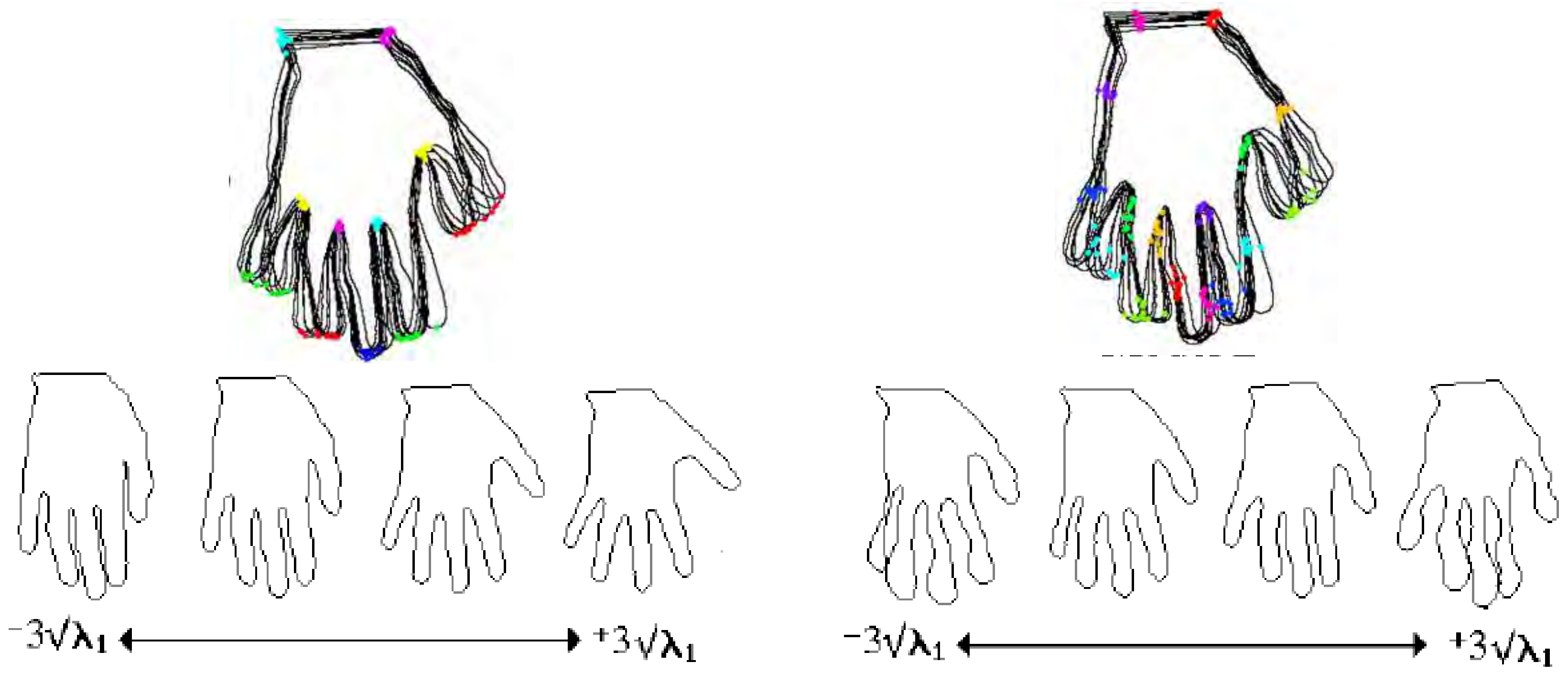
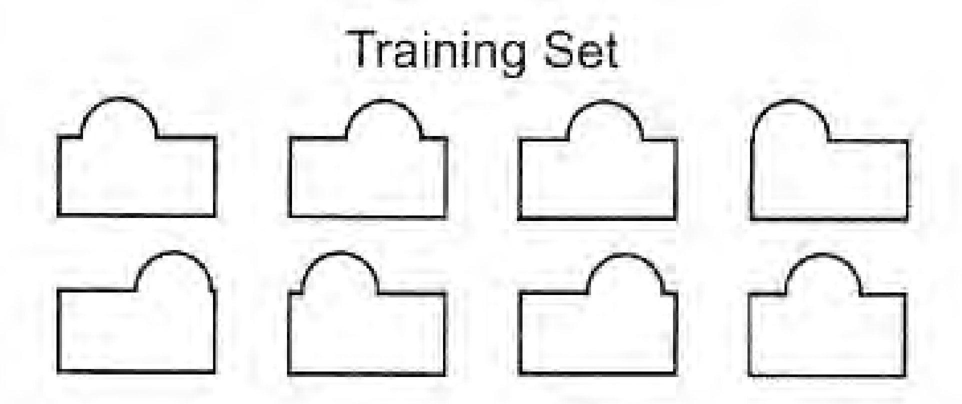


Figure 3.4. The first mode of variation of models A and B. The first parameter (b_1) is varied by $\pm 3\sqrt{\lambda_m}$.

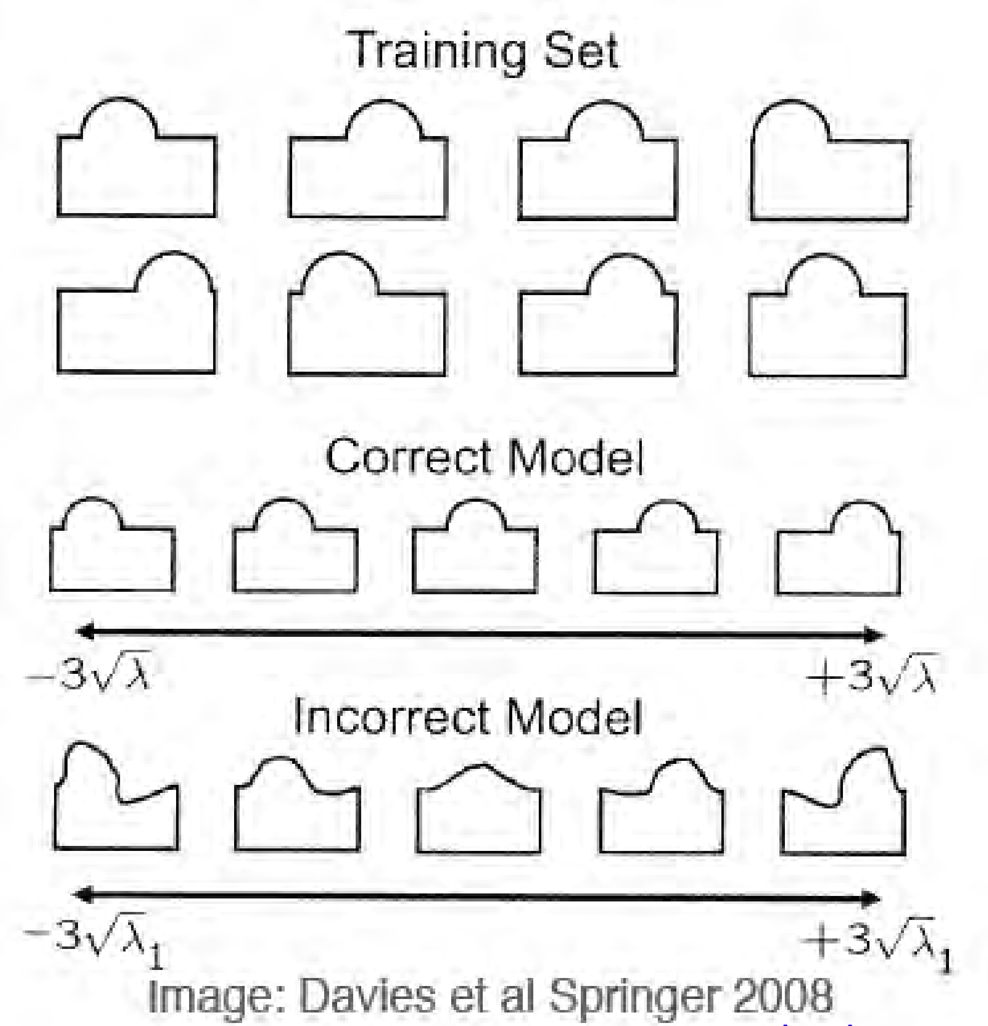
From: PhD thesis Rhodri Davies

Optimization of Correspondence: Reparametrization



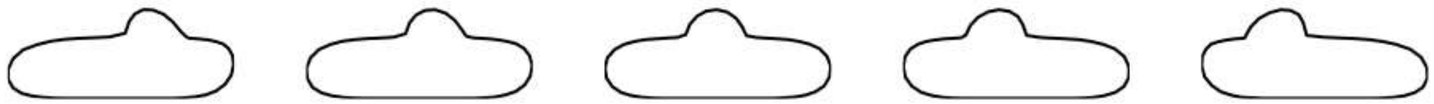
Rhodri Davies, 2000

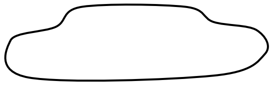
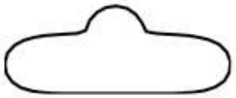
Optimization of Correspondence: Reparametrization



Rhodri Davies, 2000, Springer 2008

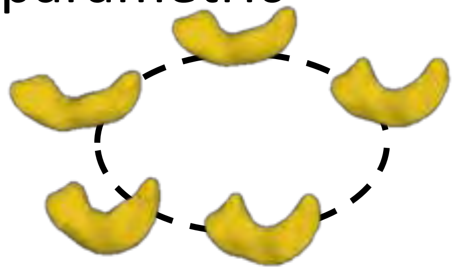
Correspondence Depends on the Population



- Image warping based on local/nearest differences 
- Alternative: take into account the trends in the ensemble 
  - Davies et al. 2000 (MDL)
  - Particle entropy (Whitaker, Cates, 2011,12)
  - Unbiased atlas building (Joshi, Davis, 2004)

Group-wise Approaches

- Use whole set of objects to determine correspondence via optimal group stats
  - Can be applied both to parametric & non-parametric descriptions



- Advantages:
  - No template bias
  - Represent all objects in a population, not just those close to the mean
  - Expect higher reliability, lower variance
  - Expect higher statistical sensitivity

Correspondence as Optimization

- Pairwise mapping of curves
- Search space: all feasible correspondences
- Objective function on quality of correspondence

- Use trend of ensemble: Optimize over population in shape space.
- Re-parameterisation function for each shape.
  - valid correspondences ⇒ diffeomorphic mapping

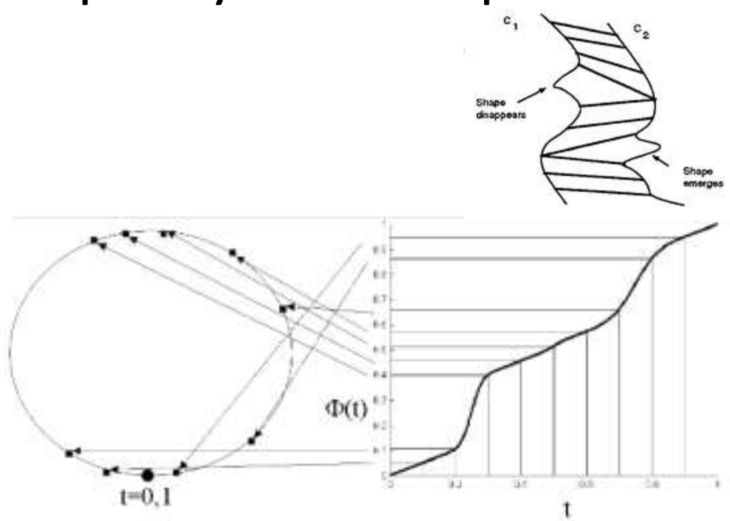
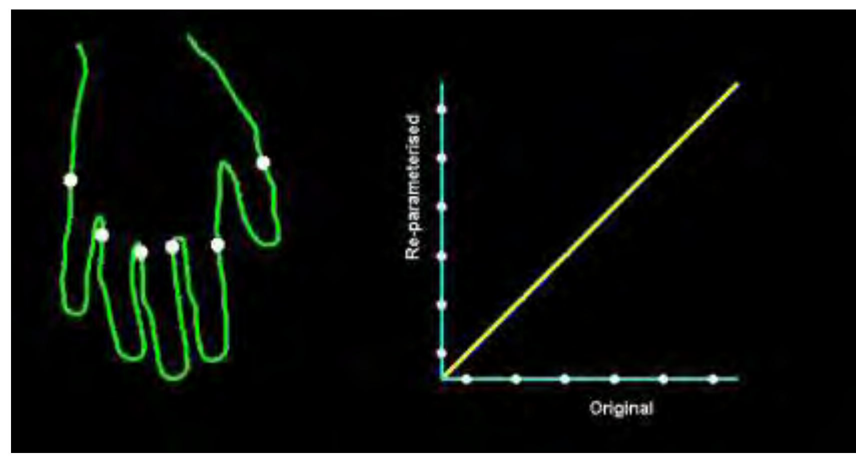


Figure 2: How a shape is sampled according to its parameterisation. The sampled points depend on the shape of the parameterisation function, Φ



Hemant Tagare, IPMI 1997

Rhodri Davies, Chris Taylor, MDL, PMI 2003

MDL: The Objective Function

- **Simplest Model has minimum stochastic complexity → Information Theory**
- **Minimum Description Length (MDL)**
- Transmit training set as encoded message
  - parameters of model, encoded data
- $$L(\Delta) \approx \sum_m \log \sigma_m + f(\sigma_n, \Delta)$$

σ_j^2 variance in j^{th} direction $\quad \sigma_m^2 \geq \Delta$

Δ lower bound on modelled variance

$f(.)$ small variance function $\quad \sigma_n^2 < \Delta$

- Use approximation to initialise

Rhodri Davies, Chris Taylor, MDL, IPMI 2003

Ensemble Correspondence: Evaluation Criteria

- **Generalization: Ability to describe instances outside of the training set**
  - leave-one-out
  - approximation error

$$G(M) = \frac{1}{n_s} \sum_{i} | \boldsymbol{x}_i - \boldsymbol{x}_i^{/}(M) |^2$$

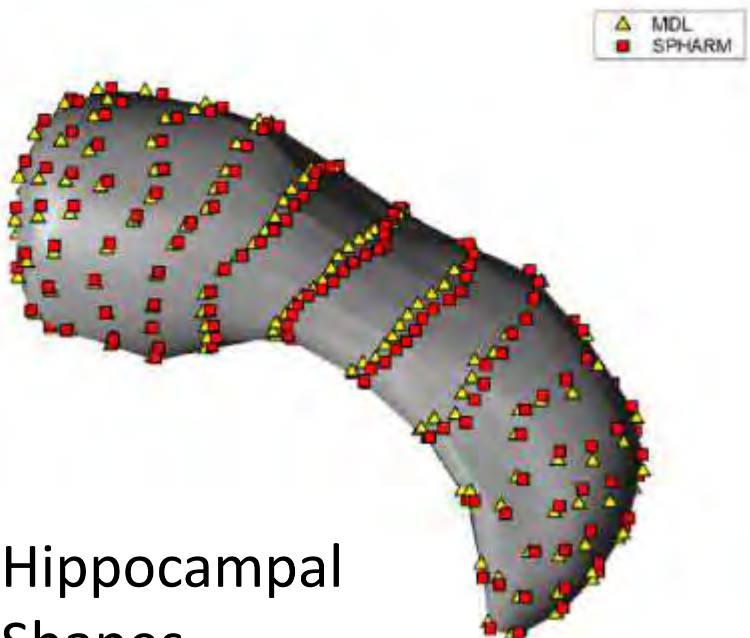
- **Specificity: Ability to represent only valid instances of the object**
  - generate new sample
  - distance to nearest training member

$$S(M) = \frac{1}{N} \sum_{j}^{N} | \boldsymbol{x}_j - \boldsymbol{x}_j^{/}(M) |^2$$

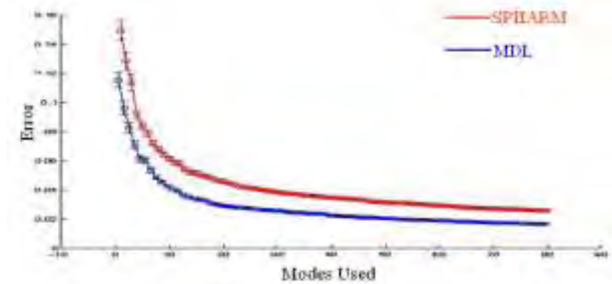
- **Compactness: Ability to use a minimal set of parameters**
  - cumulative variance

$$C(M) = \sum_{m}^{M} \lambda^{m}$$

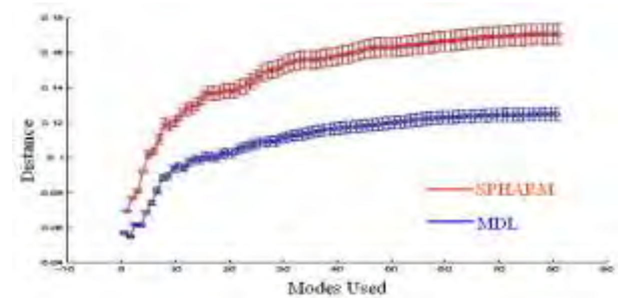
Evaluation: MDL vs. SPHARM



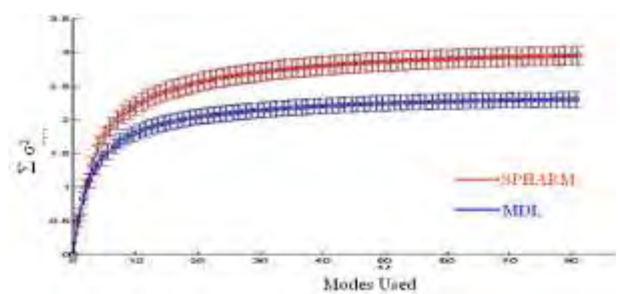
Hippocampal
Shapes
82 samples



Generalization: Leave one out



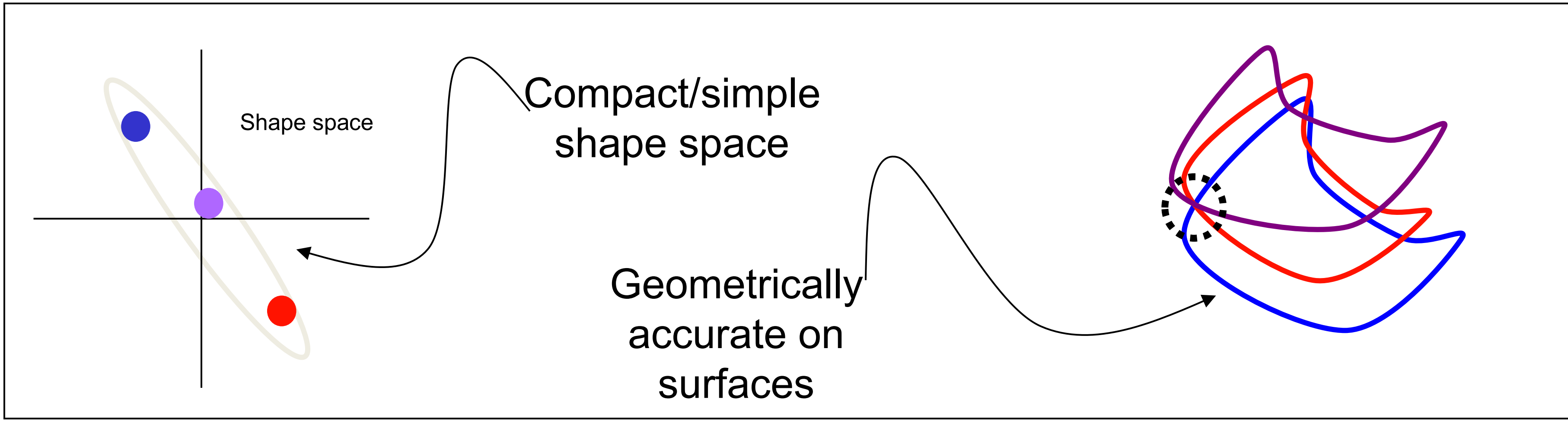
Specificity: Generate new samples, distance to nearest member

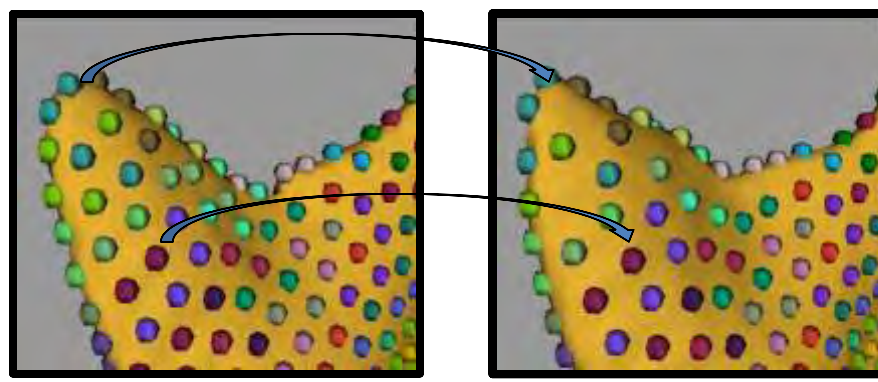


Compactness: Cumulative Variance

Rhodri Davies, Chris Taylor, MDL, PMI 2003
Styner.,.., Davies, IPMI 2003

Modeling a Shape Ensemble: Strategy for Landmark Placement





R. Whitaker, J. Cates, Uah

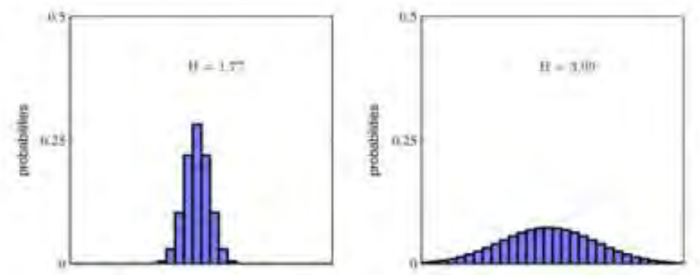
Particle-Based Shape Correspondences

- Shapes as a set of interacting particle systems
- Compact models, but balanced against geometric accuracy (good, adaptive samplings)
- Optimize *particle positions by minimizing an entropy cost function*

$$Q = H(Z) - \sum_k H(P^k)$$

Entropy of the shape ensemble

Entropy of each individual shape sampling



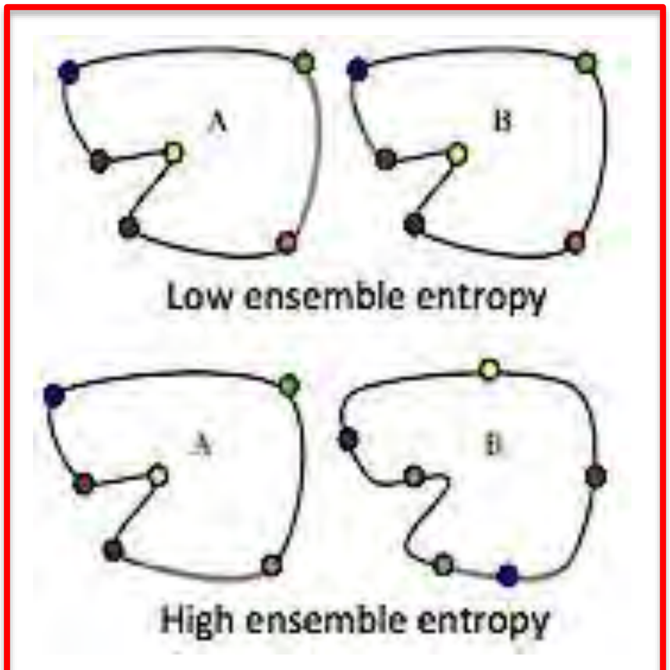
Low entropy High entropy

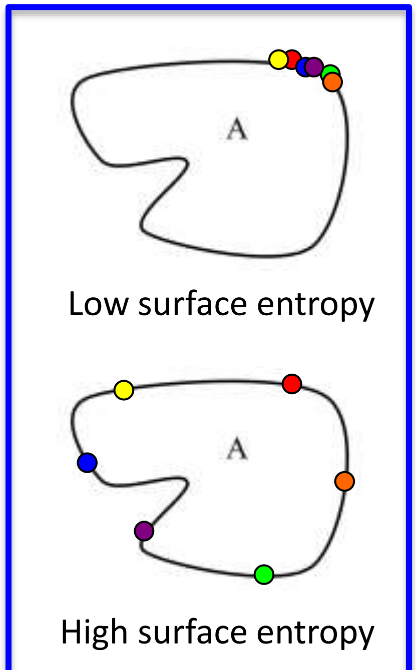
$$H[\mathbf{X}] \approx \frac{1}{2} \ln |\Sigma| = \frac{1}{2} \sum \ln \lambda$$

R. Whitaker, J. Cates, M. Datar, IPMI 2007, MICCAI 2013

Entropy-based Particle Systems

- Surfaces are discrete point sets, no parameterization
- Dynamic particles, positions optimize the information of the system: ensemble entropy, surface entropy





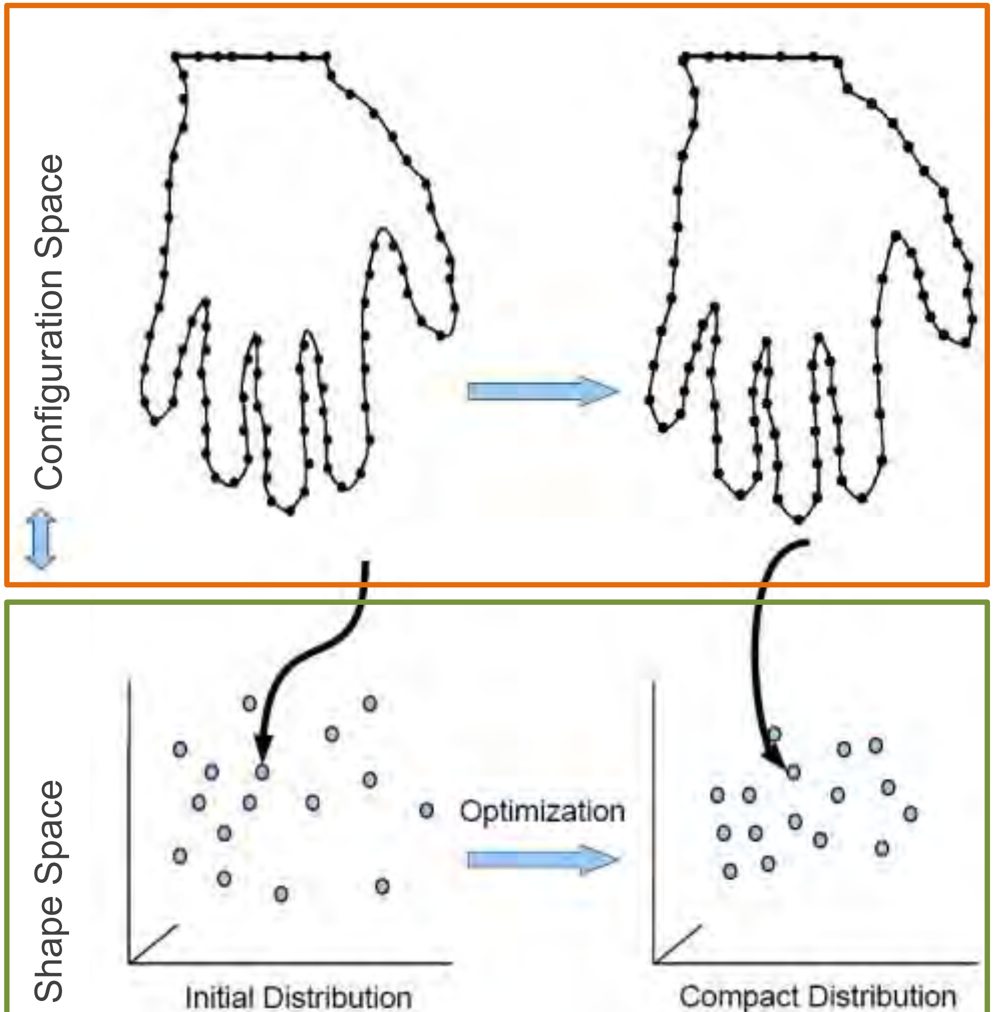
$$Q = H(Z) - \sum_k H(P^k)$$

low is better

high is better

Images: Oguz, 2009

Particle Correspondence Model



Accurate Representation
(in Configuration Space)

vs.

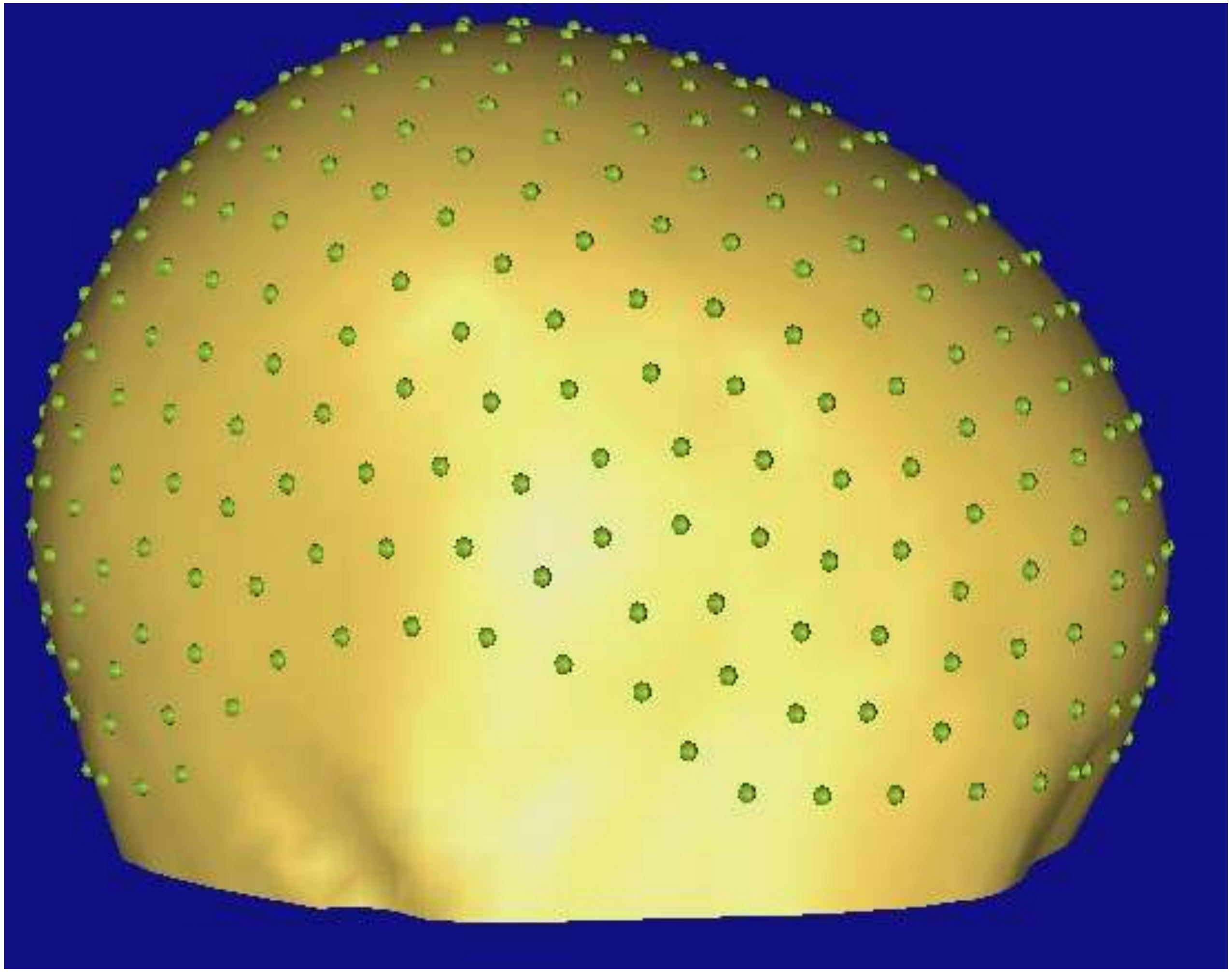
Compact Model
(in Shape Space)

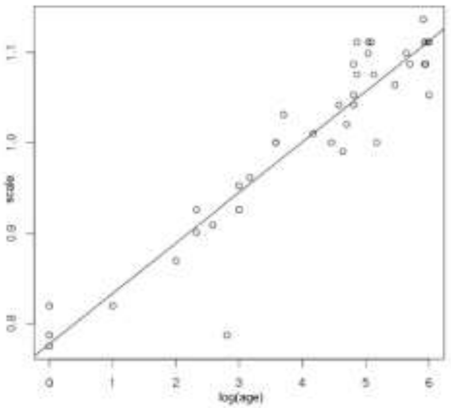
$$Q = H(\mathbf{Z}) - \sum_k H(P^k)$$

Ensemble Entropy **Surface Entropy**

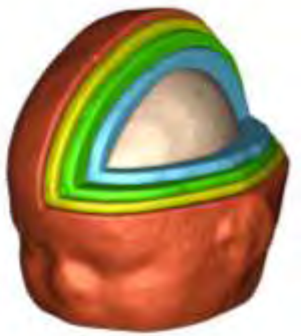
Cates et al. IPMI 2007, Datar MICCAI 2011

Modeling Head Shape Change





Changes in head size with age



Changes in head shape with age

Datar, Cates, Fletcher, Gouttard, Gerig, Whitaker, Particle-based Shape Regression, MICCAI 2009

Box-Bump

Comparison with MDL

- 24 shapes
- MDL: 128 nodes, mode 2, parameters at default*
- Particle: 100 particles per shape

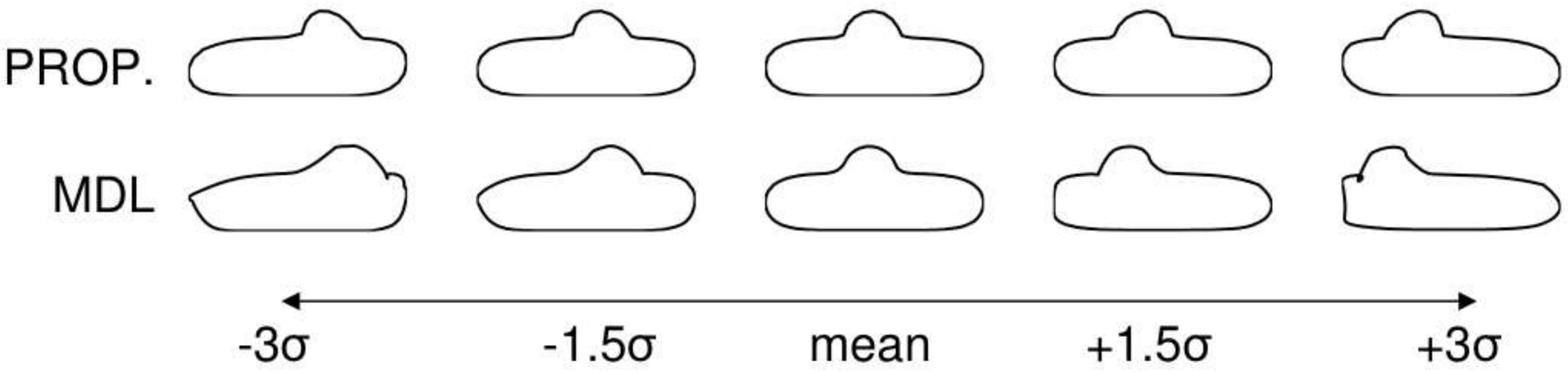
* See Thodberg, IPMI 2003 for details

Results

Single major mode of variation

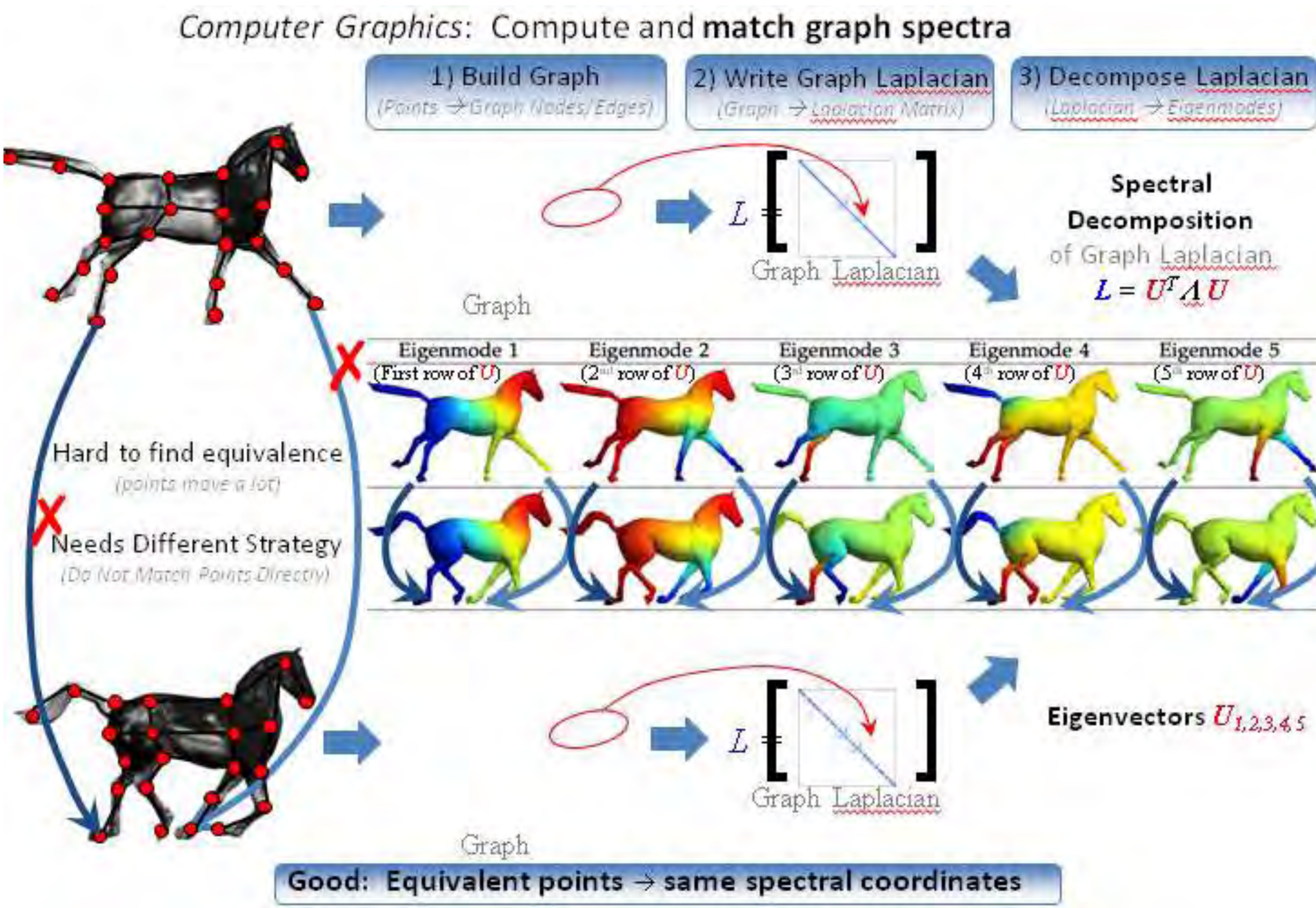
MDL: 0.34% "leakage" of total variation to minor modes

Particle: 0.0015% leakage



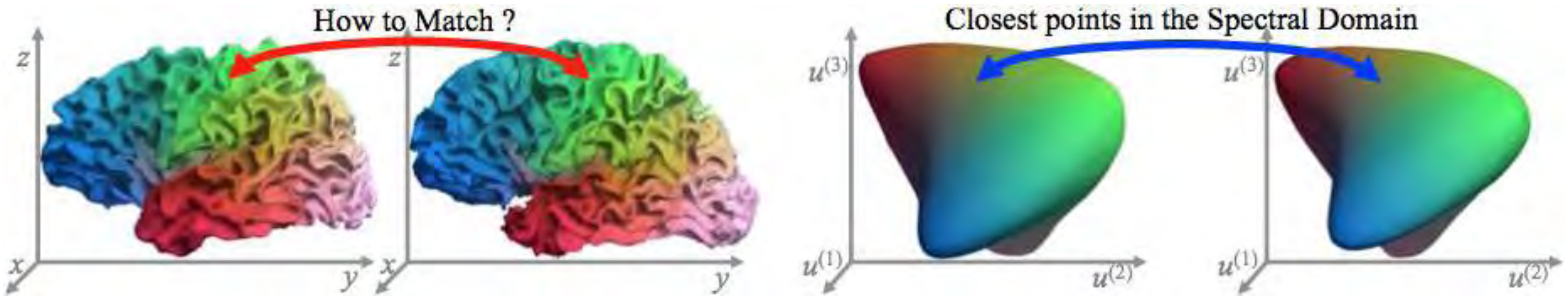
Cates & Whitaker, IPMI 2007

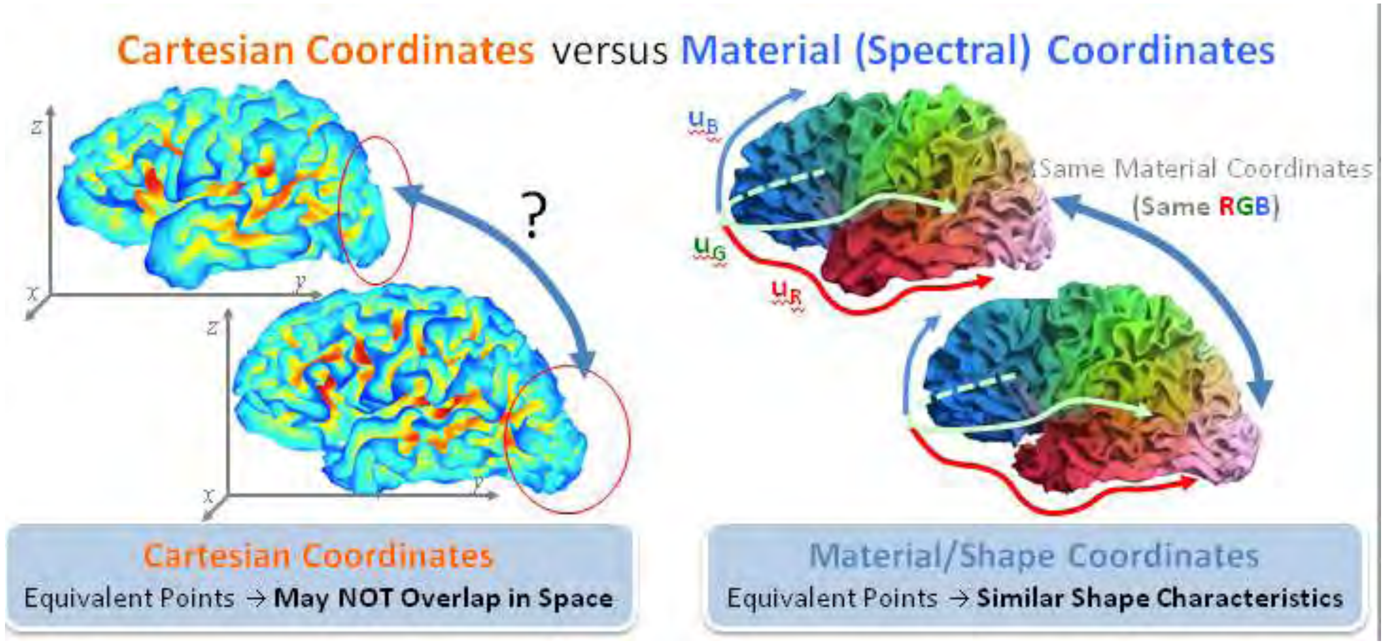
Graph Spectra/Laplacian



Courtesy ***Hervé Lombaert****, Jon Sporring, Kaleem Siddiqi, Mc Gill, IPMI 2013*

Graph Spectra/Laplacian





Courtesy ***Hervé Lombaert***, *Jon Sporring, Kaleem Siddiqi, Mc Gill, IPMI 2013*

Considering Appearance: Eigenfaces

- Very few 100x100 vectors correspond to valid face images

- model the subspace (`manifold') of face images

Sirovich & Kirby 87, Turk & Pentland 91
Source: Iasonas Kokkinos, IPAM-UCLA Course 2013

Eigenfaces

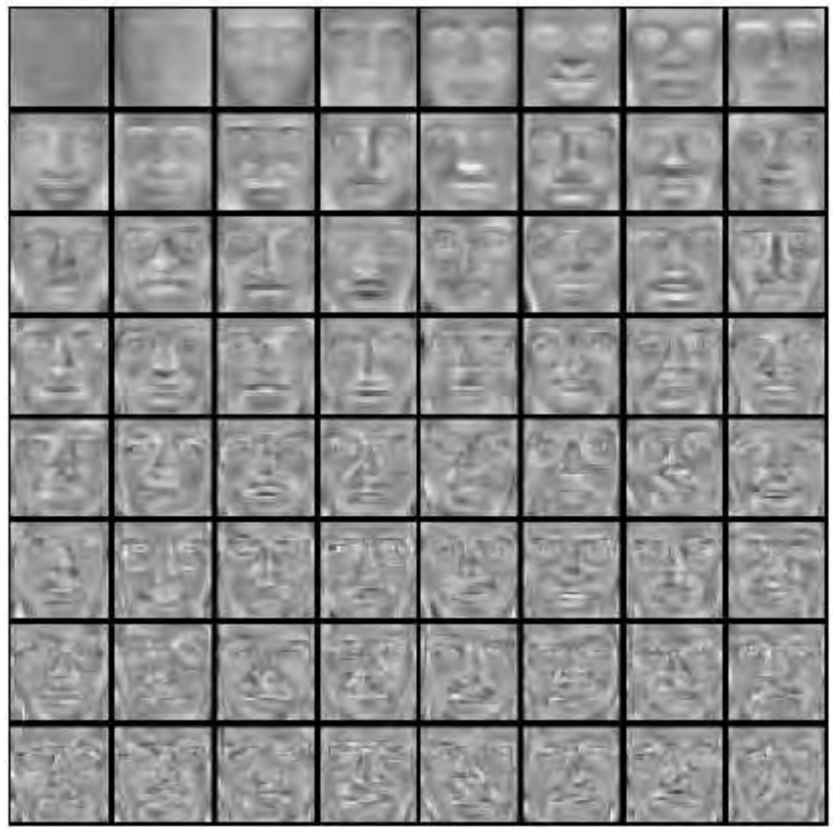
- Training images
- $x_1, \ldots, x_N$

Sirovich & Kirby 87, Turk & Pentland 91
Source: Iasonas Kokkinos, IPAM-UCLA Course 2013

Eigenfaces

Mean: μ

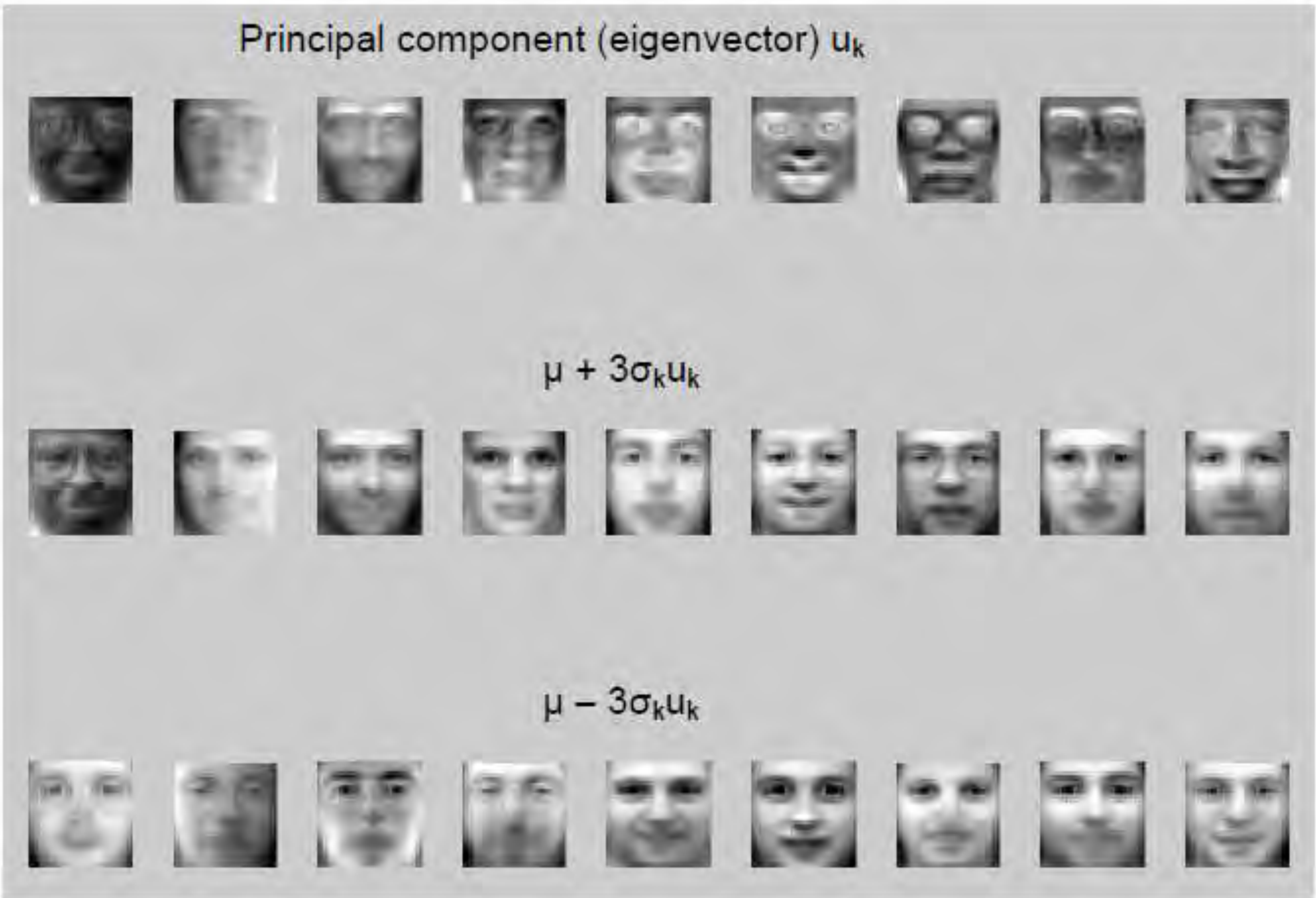
Top eigenvectors: $u_1, \ldots u_k$



Sirovich & Kirby 87, Turk & Pentland 91

Source: Iasonas Kokkinos, IPAM-UCLA Course 2013

Eigenfaces



Sirovich & Kirby 87, Turk & Pentland 91
Source: Iasonas Kokkinos, IPAM-UCLA Course 2013

Eigenfaces

- Face x in “face space” coordinates:

$$\mathbf{x} \rightarrow [\mathbf{u}_1^{\mathrm{T}}(\mathbf{x} - \mu), \ldots, \mathbf{u}_k^{\mathrm{T}}(\mathbf{x} - \mu)]$$
$$= \quad w_1, \ldots, w_k$$

- Reconstruction:

 =  + 

$\hat{x} \;=\; \mu \;+\; w_1u_1+w_2u_2+w_3u_3+w_4u_4+ \ldots$

Sirovich & Kirby 87, Turk & Pentland 91
Source: Iasonas Kokkinos, IPAM-UCLA Course 2013

Active Shape and Appearance Models

- Statistical models of **shape *and* texture**
- Generative models
  - general
  - specific
  - compact (~100 params)

Courtesy of Chris Taylor, 1995

Building an Appearance Model

- Labelled training images
  - landmarks represent correspondences

Courtesy of Chris Taylor, 1995

Building an Appearance Model

- For each example

Shape: $\mathbf{x} = (x_1, y_1, \ldots, x_n, y_n)^T$

Warp to mean shape

Raster Scan

Texture: $\mathbf{g}$

Courtesy of Chris Taylor, 1995

Building an Appearance Model

- Principal component analysis
  - shape model: $\mathbf{x} = \bar{\mathbf{x}} + \mathbf{P}_s \mathbf{b}_s$
  - texture model: $\mathbf{g} = \bar{\mathbf{g}} + \mathbf{P}_g \mathbf{b}_g$
- Columns of $\mathbf{P}_r$ form shape and texture bases
- Parameters $\mathbf{b}_r$ control modes of variation

Courtesy of Chris Taylor, 1995

Shape and Texture Modes

Shape variation (texture fixed)

Texture variation (shape fixed)

Courtesy of Chris Taylor, 1995

Combined Appearance Model

- Shape and texture may be correlated
  - PCA of $\begin{pmatrix} \mathbf{b}_s \\ \mathbf{b}_g \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{x} \\ \mathbf{g} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{x}} \\ \bar{\mathbf{g}} \end{pmatrix} + \begin{pmatrix} \mathbf{Q}_\mathbf{x} \\ \mathbf{Q}_\mathbf{g} \end{pmatrix} \mathbf{c}$

Varying appearance vector **c**

Courtesy of Chris Taylor, 1995

Colour Appearance Model

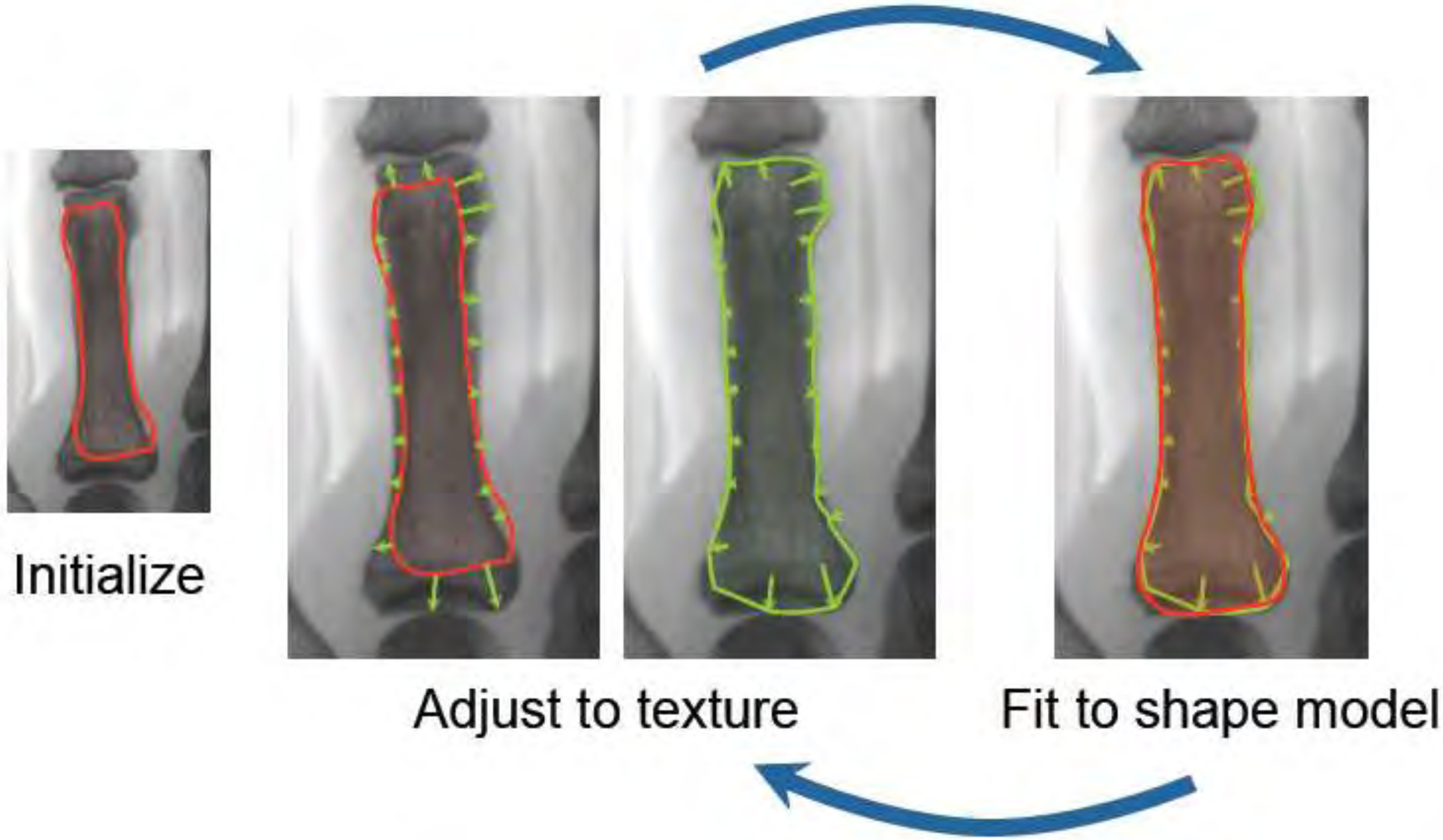
c_1

c_2

c_3

Courtesy of Chris Taylor, 1995

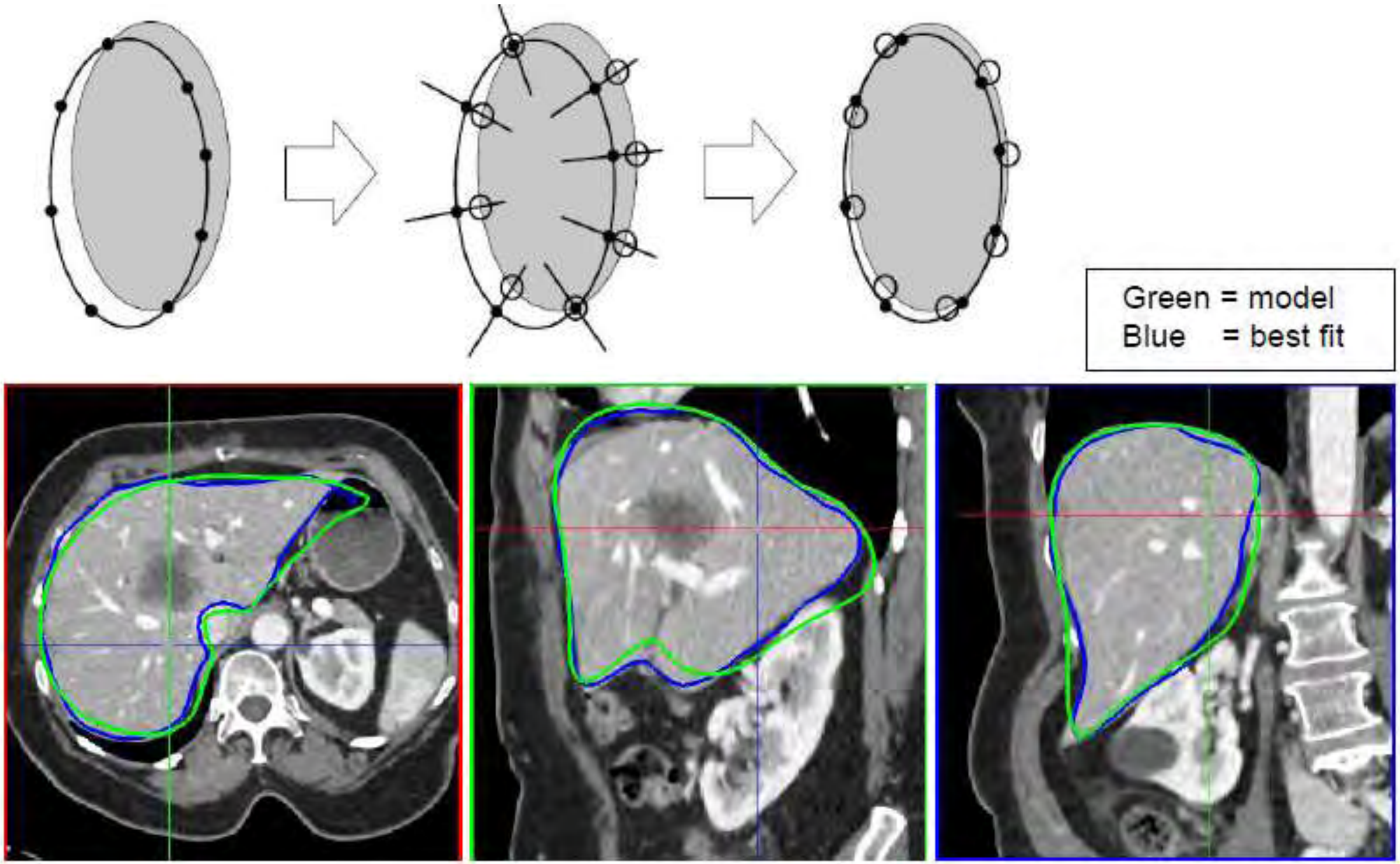
AAM Search – Deformable Automatic Segmentation



Slide Credit: G. Lang

Source: Iasonas Kokkinos, IPAM-UCLA Course 2013

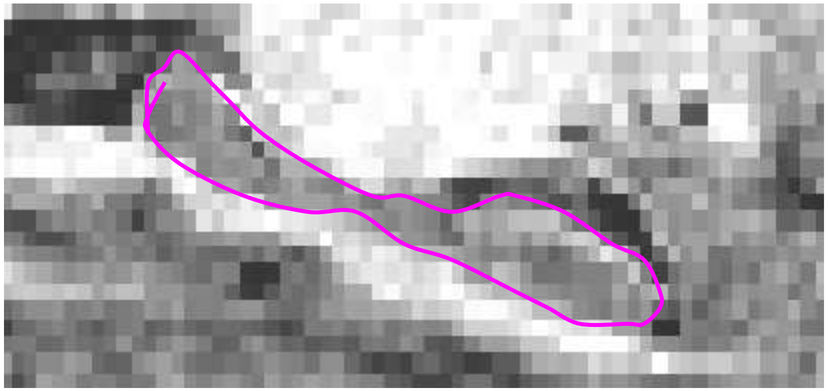
Active Shape Model Search

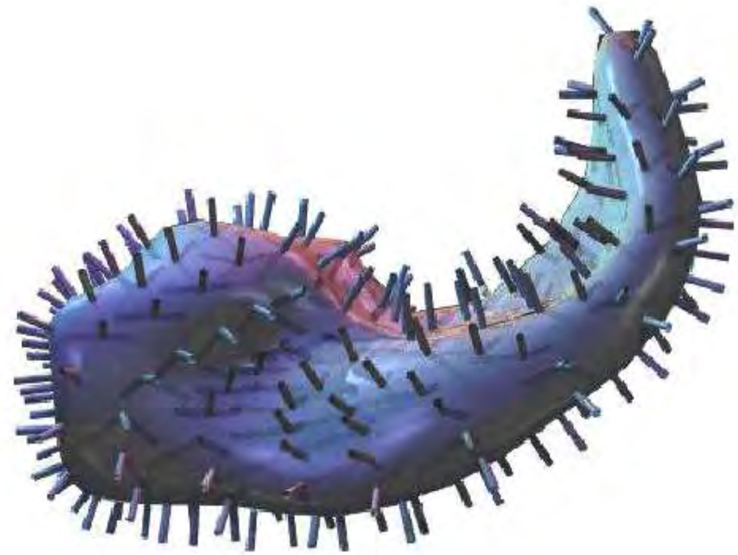


Method: Cootes et al., 1995

Slide: T. Heimann: - Shape Symposium 2014, Delémont

3D Hippocampus: ASM & AAM Modeling for Deformable Segmentation

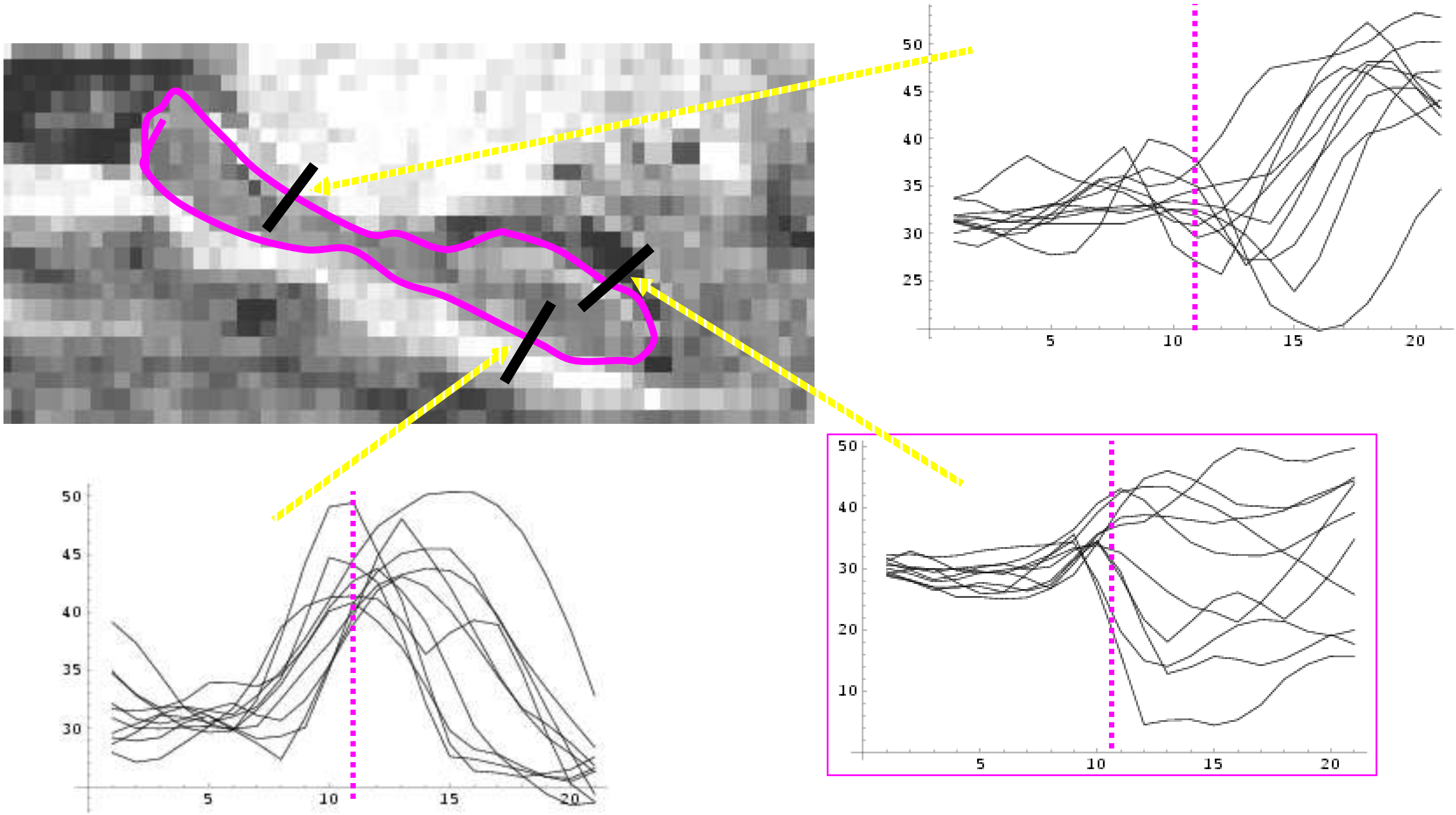




Profiles normal to surface capture local image intensity function (hedgehog)

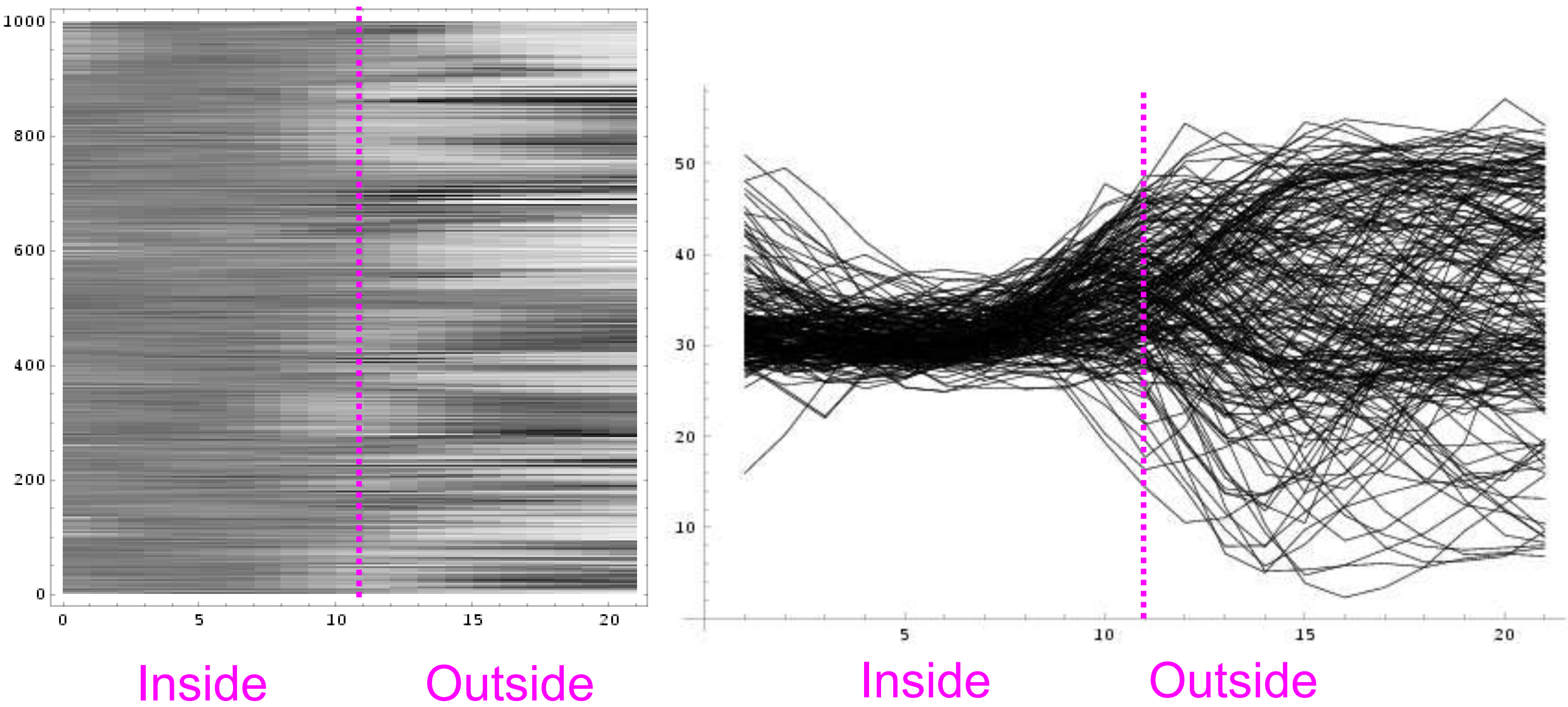
Szekely, Kelemen, Brechbuehler, Gerig, MedIA 1996
Styner & Gerig, MedIA 2004

Appearance Profiles across 10 training images



Styner & Gerig, UNC

Appearance Profiles

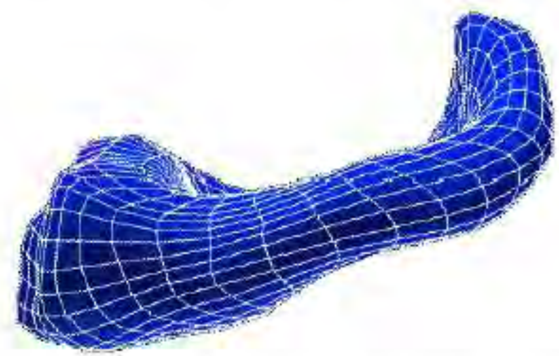


Styner & Gerig, UNC

Dual shape representations: PDM/SPHARM

Surface Points:

Spherical Harmonics:



$$\mathbf{x} = \mathbf{A}\mathbf{c}$$

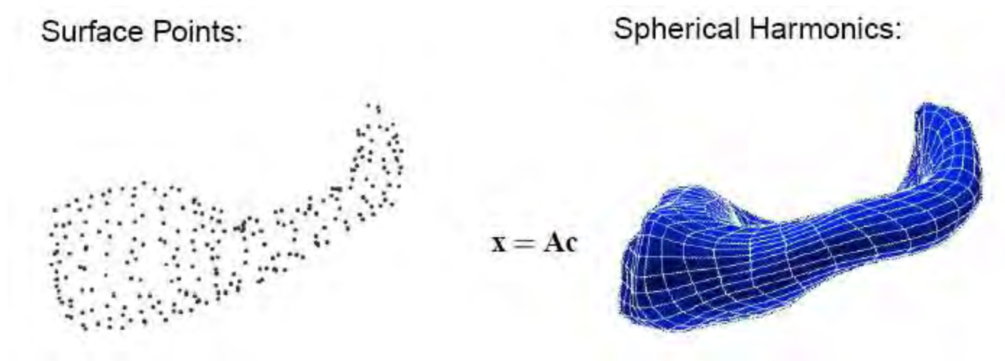
Local Description
Regular Sampling
Position of Profiles

Global Shape Description
Finding Correspondence
Computing Surface Normals

$$\mathbf{x}(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \mathbf{c}_l^m \, Y_l^m(\theta, \phi)$$

Styner & Gerig, UNC

Computing the fit



Parameter Space:

Statistics in spher. harm.: $$\mathbf{c} = \bar{\mathbf{c}} + \mathbf{P_c b}$$

Multiplying by $\mathbf{A}$: $$\mathbf{Ac} = \mathbf{A}\bar{\mathbf{c}} + \mathbf{AP_c b}$$

Object Space:

Statistics in coordinates: $$\mathbf{x} = \bar{\mathbf{x}} + \mathbf{P_x b}$$

Altering coordinates with $d\mathbf{x}$: $$(\mathbf{x} + d\mathbf{x}) = \bar{\mathbf{x}} + \mathbf{P_x}(\mathbf{b} + d\mathbf{b})$$

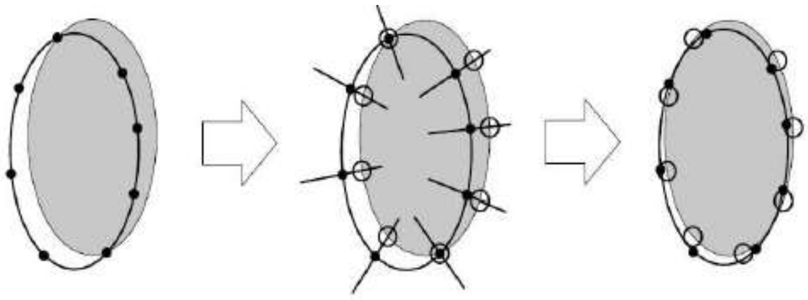
Set of eq. to solve: $$d\mathbf{x} = \mathbf{P_x} d\mathbf{b}$$

Szekely, Kelemen, Styner & Gerig, ETH, UNC

Deformation Forces and Constraints

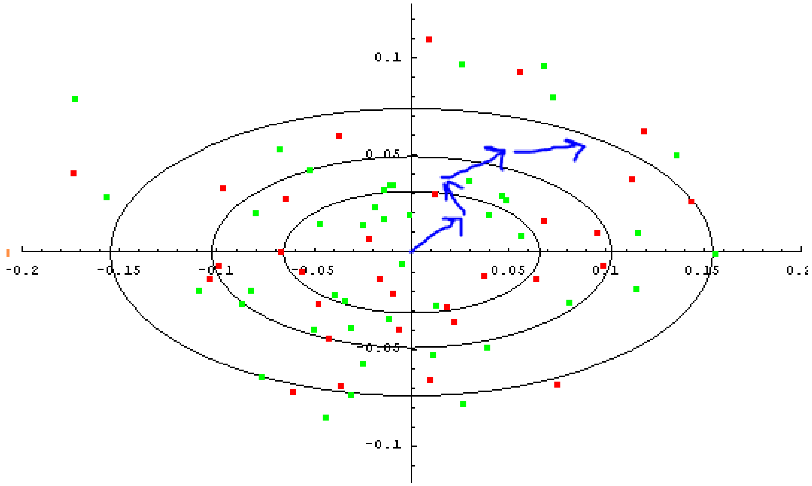
Driving deformation force at boundary points x_i:

- Start with mean shape.
- Correlate local boundary appearance with statistical model.
- Find suggested "shift" for each point $x_i \rightarrow dx_i$.
- Convert dx into shift in shape space $d_b \rightarrow$ shape

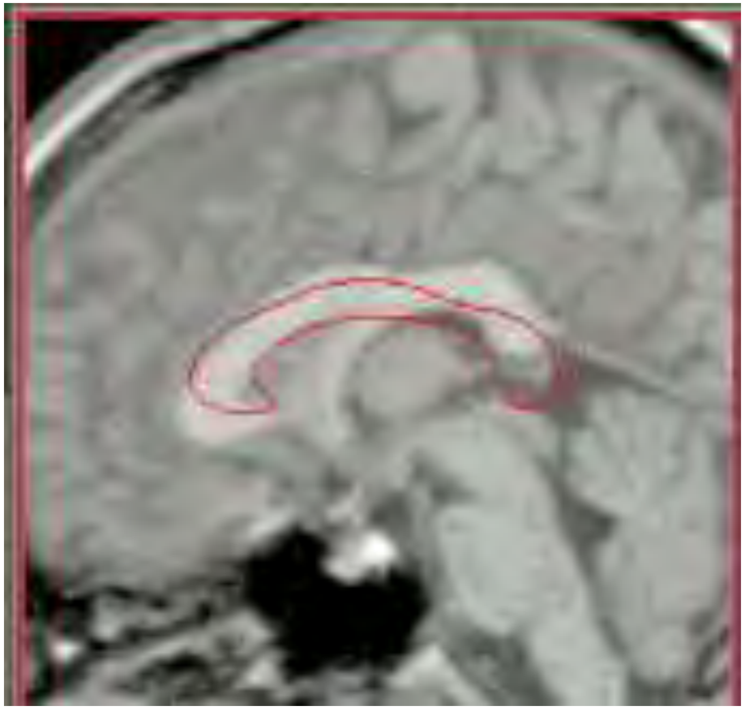


Shape constraints:

- Ensure that $d + d_b$ stays within predefined Mahalanobis distance of shape space.



Deformable Model Segmentation



Segmentation of corpus callosum via deformable model segmentation, max order 10 (40 coeffs)

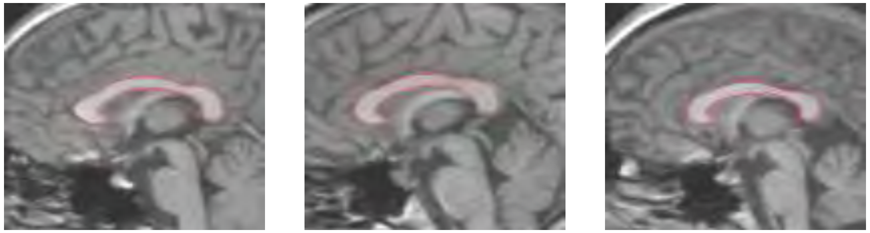


Fig. 1: Visualization of 3 MRI mid-hemispheric slices and the final positions (in red) of the automatic corpus callosum segmentation algorithm using deformable shape models.

Fourier Descriptors of Shape Contours **Paper-Kuhl-Giardina-1982**, **Paper-Staib-Duncan-1992**

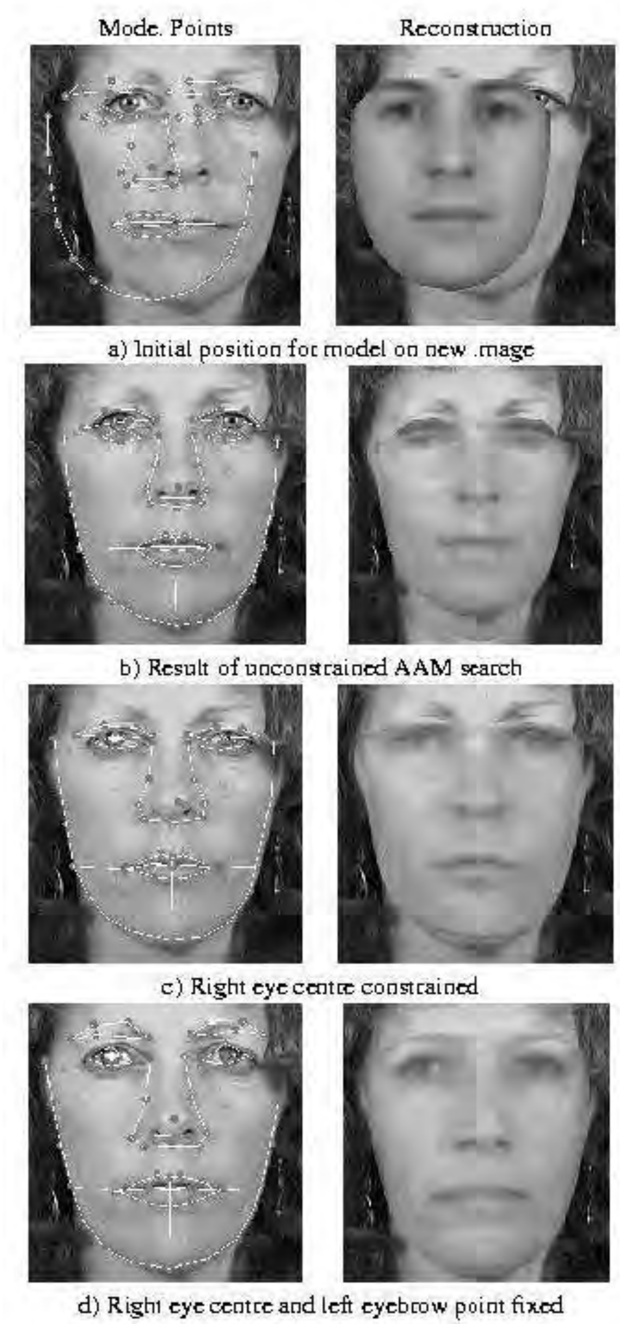
Styner & Gerig, UNC

Segmentation in Action

Styner & Gerig, UNC

Constrained AAMs

- Comparison of constrained and unconstrained AAM search
- **Conclusions**: Cannot directly handle cases well outside of the training set (e.g. occlusions, extremely deformable objects)



Courtesy of Chris Taylor

Non Unimodal Shape Space: Gaussian Mixture Model

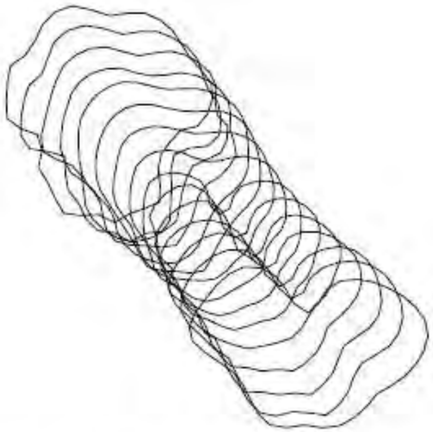


Figure 9: Contours from sequential slices

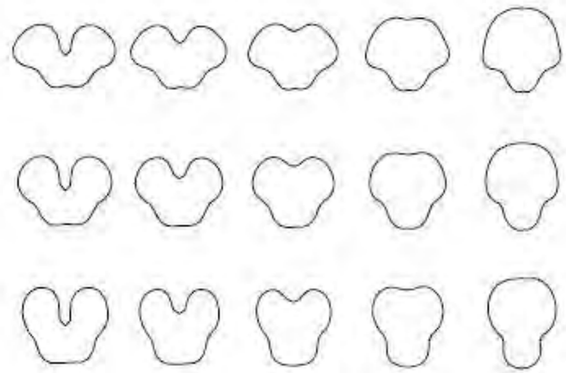


Figure 10: Shape for b_1 vs b_2 for brain stem

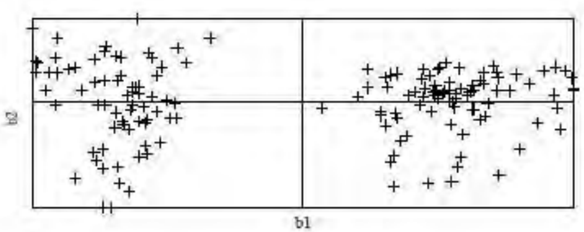


Figure 11: Plot of b_1 vs b_2 for brain stem

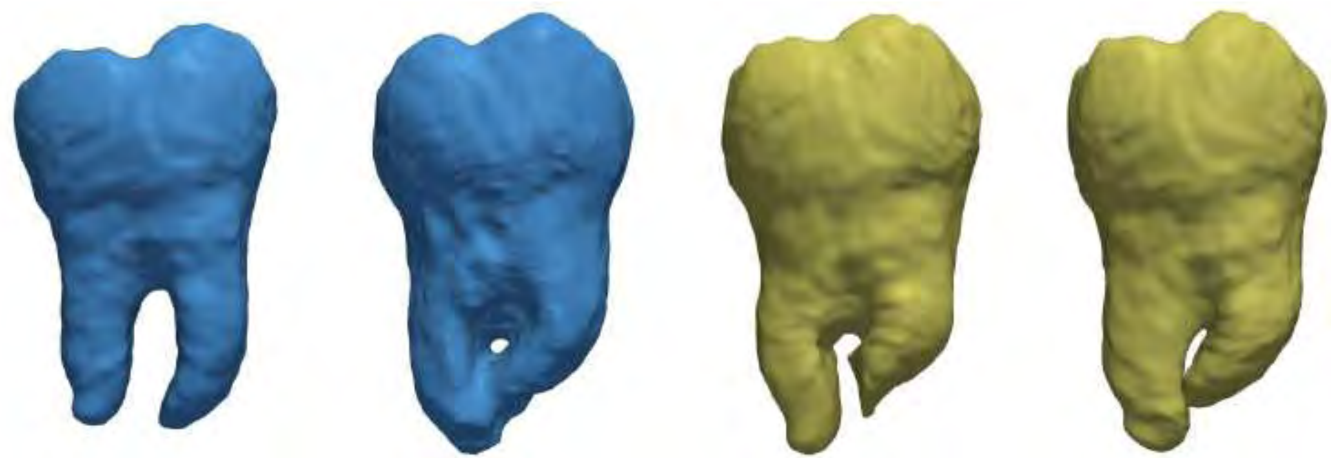
Figure 12: pdf approximation with 2 gaussians

A Mixture Model for Representing Shape Variation, Cootes et al., IVC 1999

Variations of SSM for Segmentation

Geodesically Damped Shape Models (*Christoph Jud, Thomas Vetter, 2014)*

- *SSM training … too restrictive …, new* method for model bias reduction…, achieved by damping the empirical correlations between points on the surface which are geodesically wide apart.
- Yields locally more flexibility of the model and a better overall segmentation performance.



(a) reference (b) target (c) model fit k_G (d) model fit k_{geo}

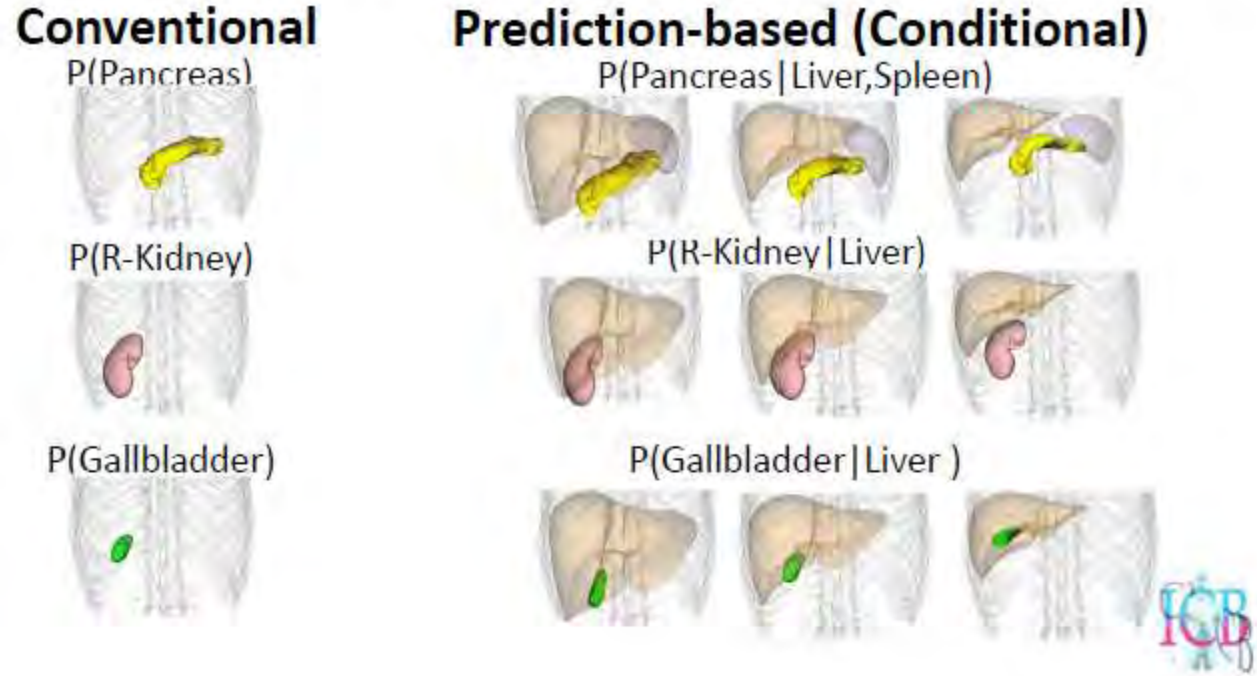
Jud et al., Proceedings, Shape Symposium 2014, Delémont

Advanced AAMs close to the Clinic

Prediction-based Statistical Atlas
Statistical Shape Model (SSM)

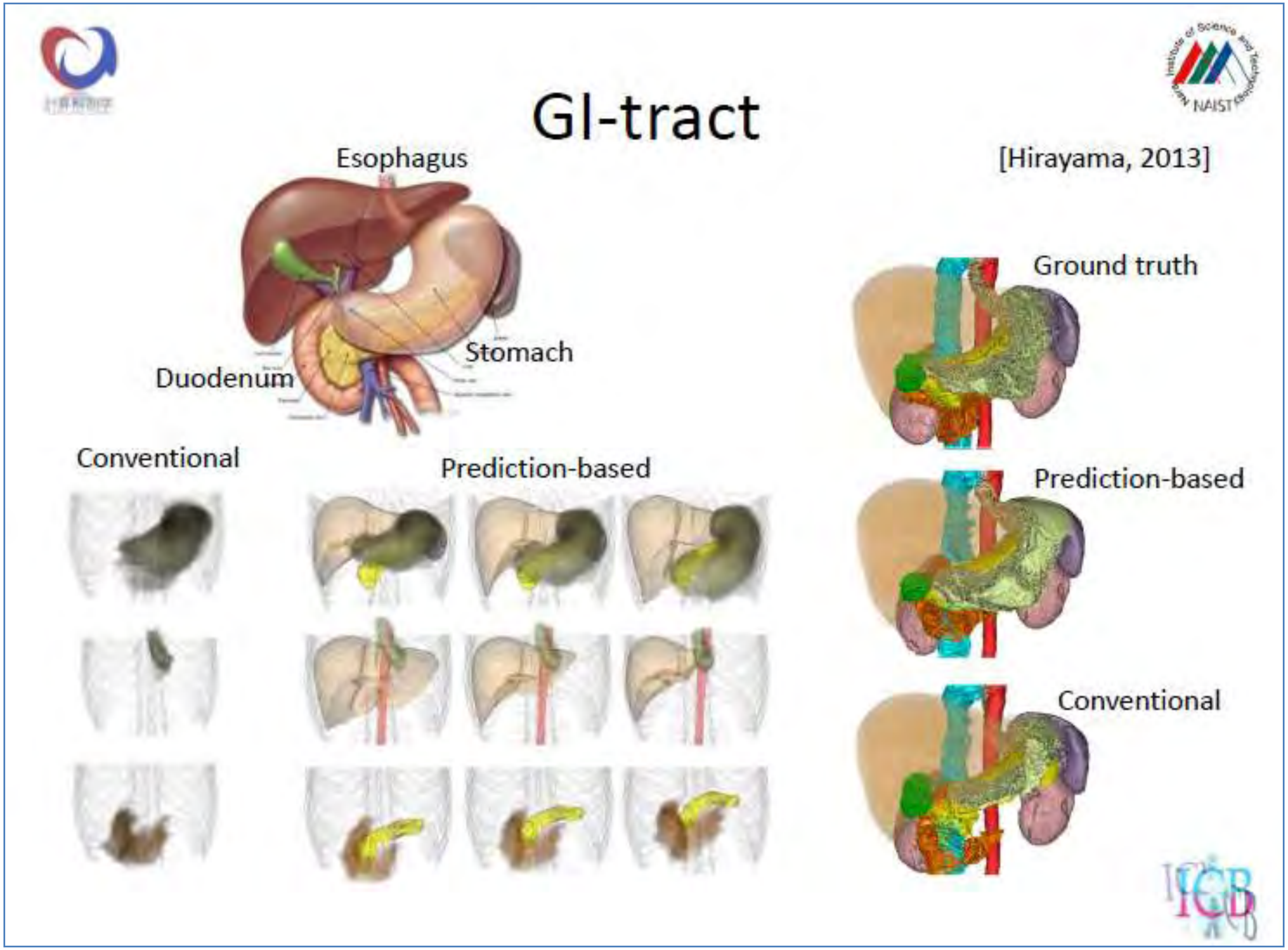
- The prediction error *E* is also modeled using PCA in prediction-based SSM to obtain more constrained variability.

  $E = S - S'$ (S: True shape, S': Predicted shape, E: Prediction error)



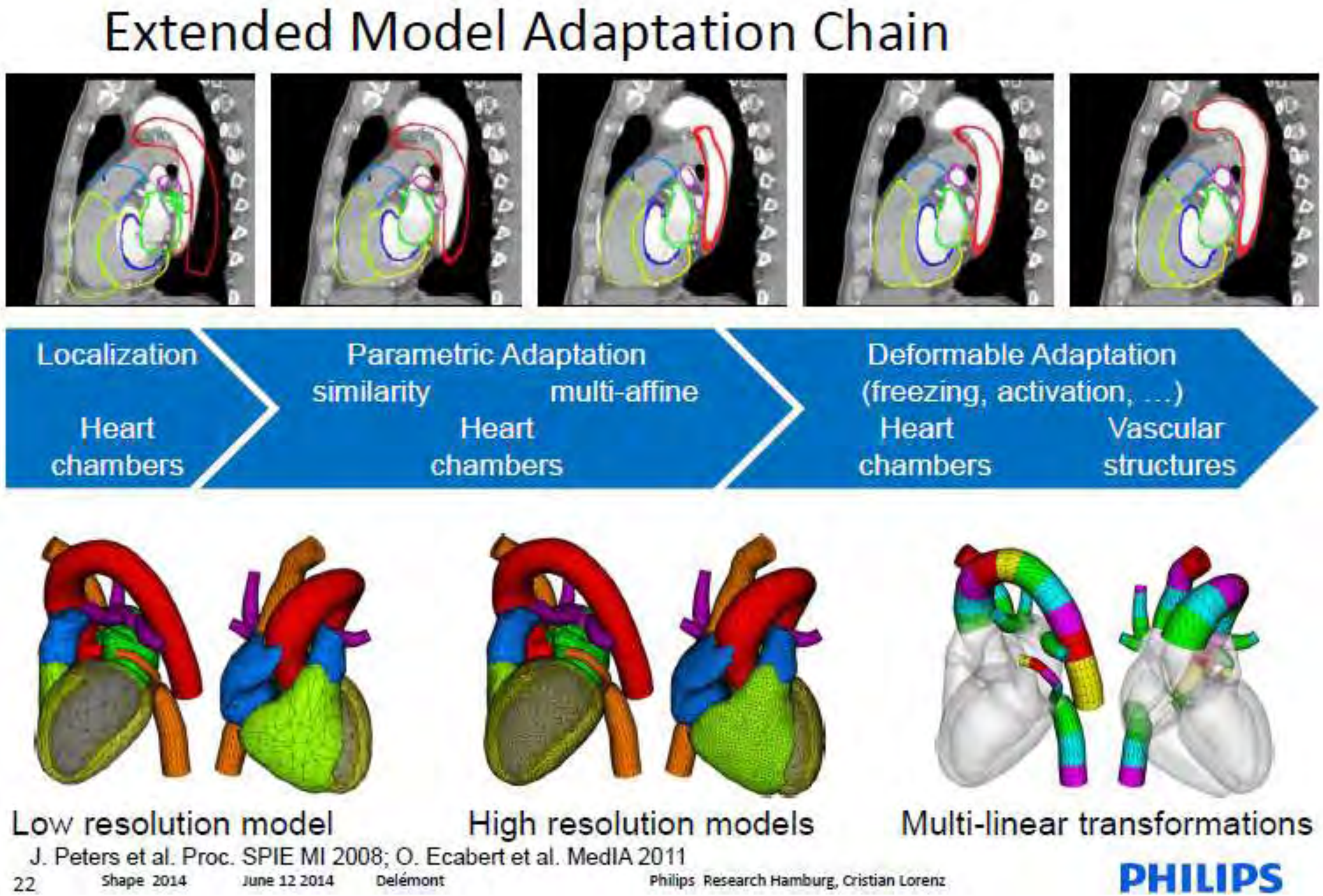
Slide: Yoshinobu Sato: - Shape Symposium 2014, Delémont

Advanced AAMs close to the Clinic



Slide: Yoshinobu Sato: - Shape Symposium 2014, Delémont

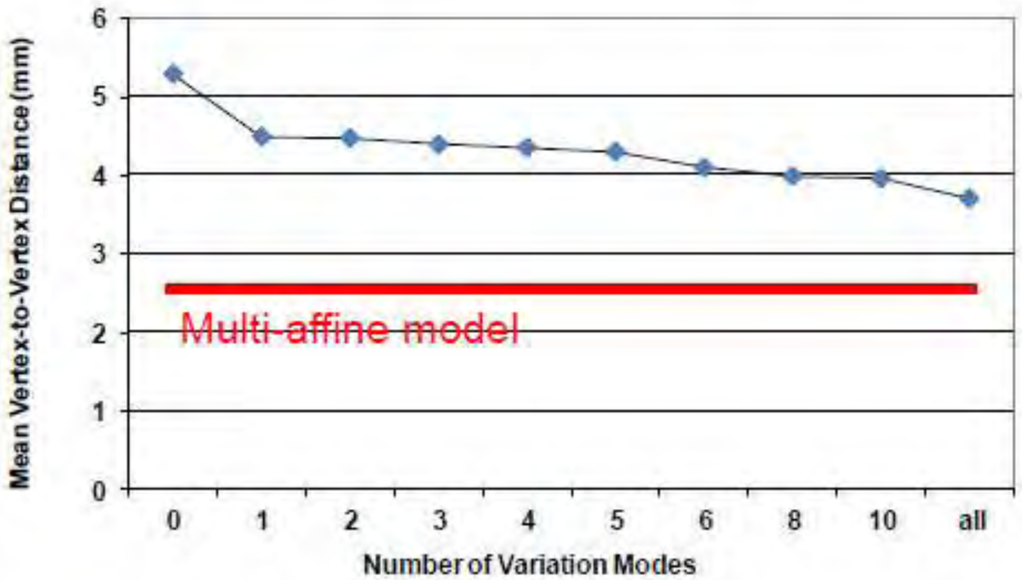
Alternative to PCA: Multi-affine



Slide: Christian Lorenzen: - Shape Symposium 2014, Delémont

Alternative to PCA: Multi-affine

- PCA/PDM model derived from 28 hearts of 13 patients.
- Approximation error (leave-one-out test).



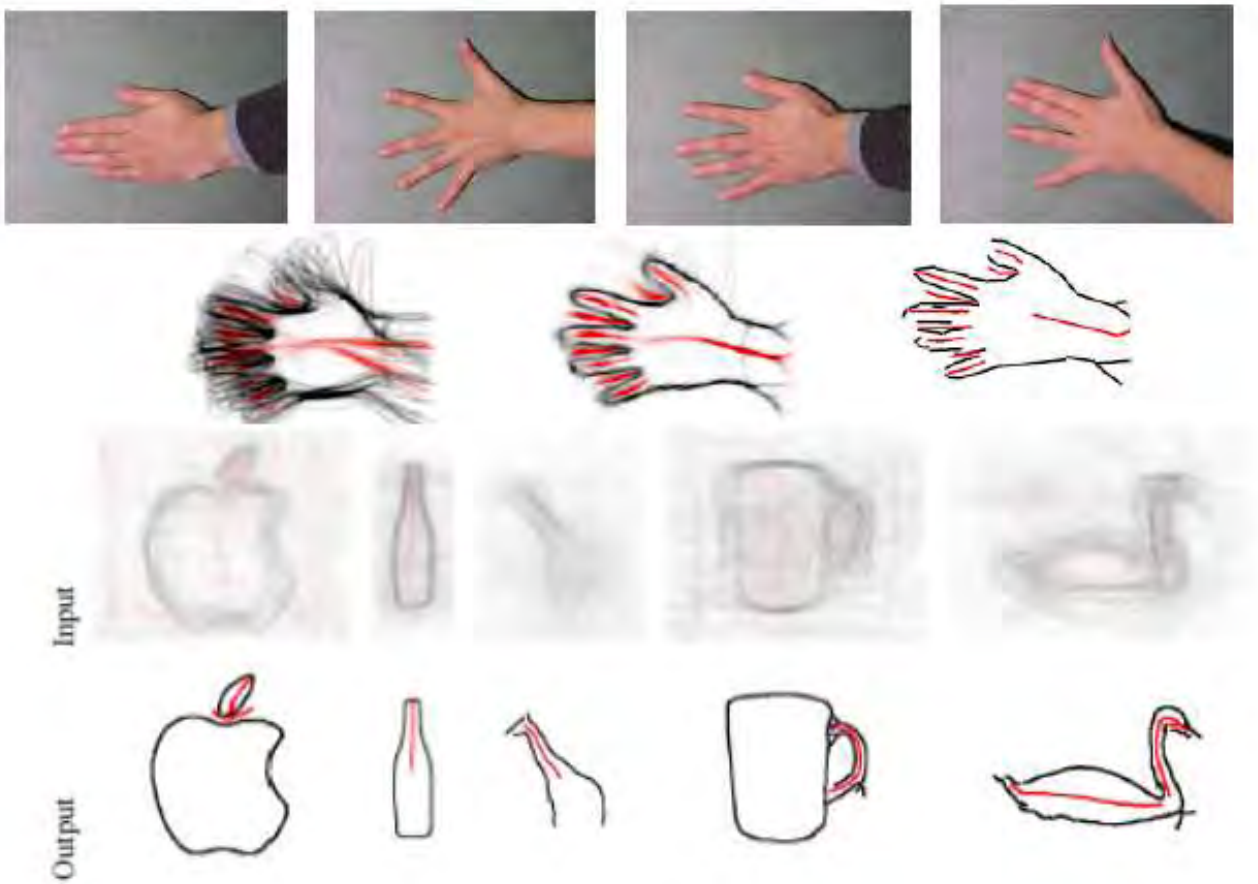
⇒ Multi-affine heart model outperforms PCA/PDM model.

O. Ecabert et al. Proc. SPIE MI 2006; O. Ecabert et al. IEEE TMI 2008

Slide: Christian Lorenzen: - Shape Symposium 2014, Delémont

EM-based AAM learning

Hand, apple, giraffe, mug, swan models (2011)



I. Kokkinos and A. Yuille, Inference and Learning for Hierarchical Shape Models, IJCV 2011

Source: Iasonas Kokkinos, IPAM-UCLA Course 2013

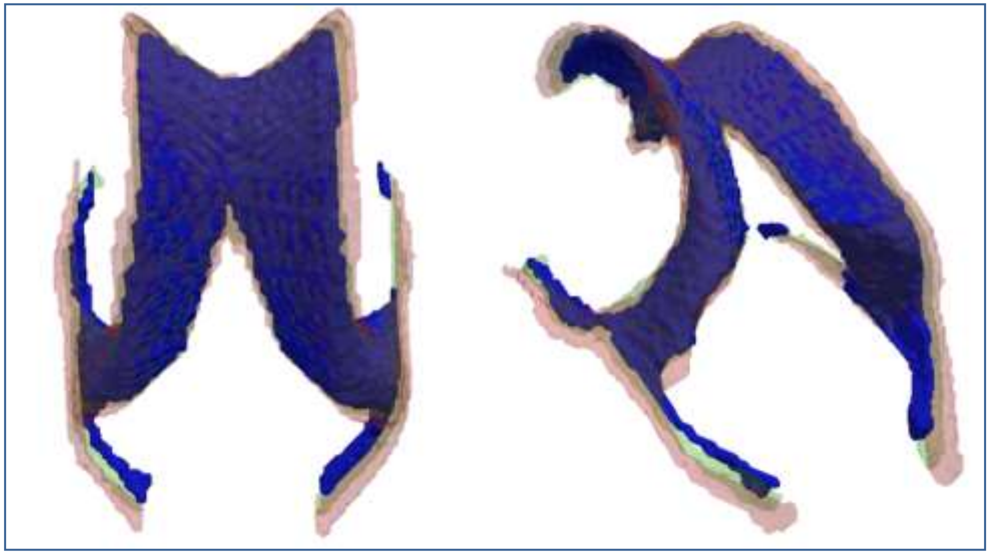
Contents

- What is Shape?
- Geometry Representations
- Kendall Shape Space
  - Statistical Shape Modeling (SSM)
  - Correspondences
  - Active Shape & Appearance Models (ASM, AAM)
- Shape Statistics via Deformations
  - Correspondence-free Mapping & Stats via “currents”
  - Ambient Space Deformations via Diffeomorphisms
  - Statistics of Deformations of Ambient Space

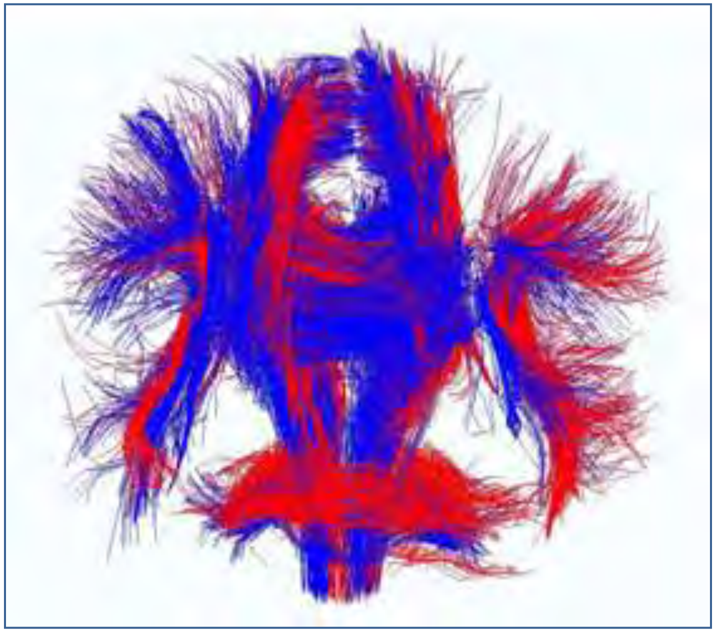
Correspondence-free Shape Analysis

Problems:

- Correspondence depends on shape parameterization
- Shapes with variable topology: correspondence undefined



Brain ventricles for infants 6mo to 2yrs



DTI Fiber Tracts from two subjects

movie

Durrleman, Pennec, Trouve, Ayache et al., IJCV 2013

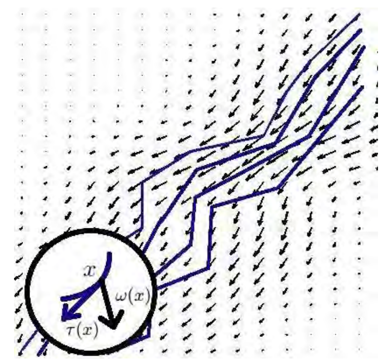
Correspondence-free similarity measures

- Depends on the kind of objects:
  - Images: sum of squared differences $\int |I(x) - I'(x)|^2 \, dx$
  - Landmarks: sum of squared differences $\sum_k |x_k - x'_k|^2$
  - Surface mesh and curves:
    - Currents [Glaunès'05]

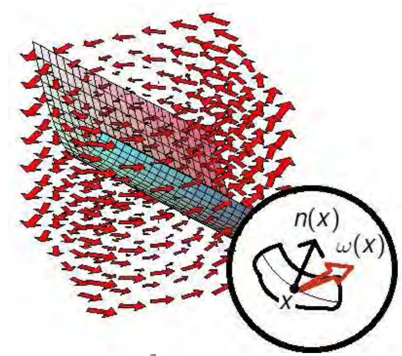
$$||F - F'|| = \sup_{||\omega||<1} |F(\omega) - F'(\omega)|$$

$$\langle F, F' \rangle = \sum_p \sum_q \exp\left(-\frac{|x_p - x'_q|^2}{\sigma_W^2} \right) \tau_p^T \tau'_q$$

τ: tangents of curves/normals of surfaces



$$F(\omega) = \sum_i \int_{C_i} \omega(x)^T \tau_i(x) d\lambda_i(x)$$



$$S(\omega) = \int_S \omega(x)^T n(x) d\sigma(x)$$

- No point correspondence needed
- Efficient numerical schemes (FFT)
- Robust to changes in topology
- Robust to differences in
  - mesh sampling
  - mesh imperfections…

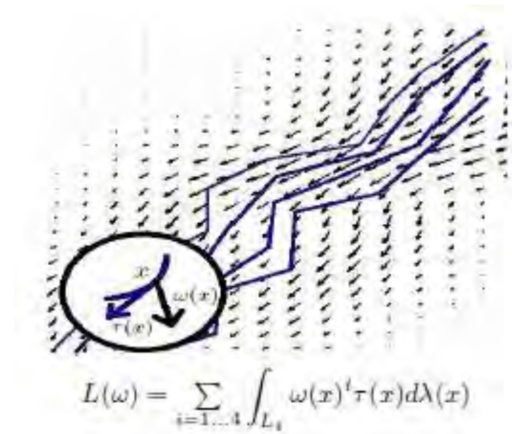
→ ***Usable routinely on large data sets***

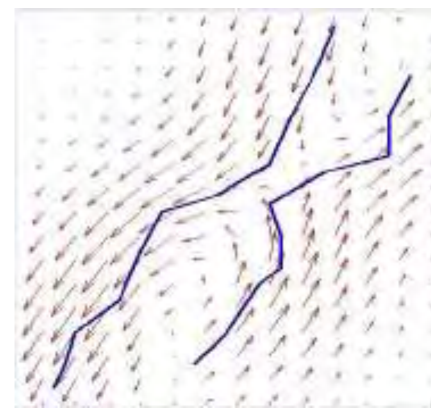
“Correspondence-free” Registration: Currents

Topology and shape differences and noise can make point-to-point correspondence hard:

- **Currents**: Objects that integrate vector fields
- **Shape**: Oriented points = Set of normals (tangents)
- **Distance** between curves:

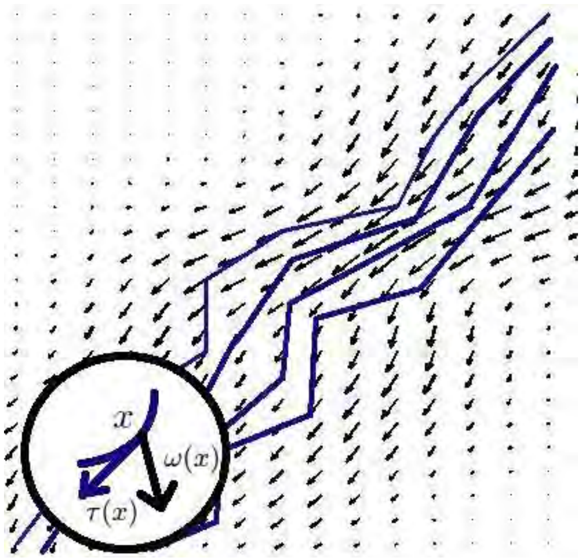
$$d(L_1, L_2)^2 = \int_{L_1} \omega_1(x)^t \tau_1(x) dx + \int_{L_2} \omega_2(x)^t \tau_2(x) dx - \int_{L_1} \omega_2(x)^t \tau_1(x) dx - \int_{L_2} \omega_1(x)^t \tau_2(x) dx$$



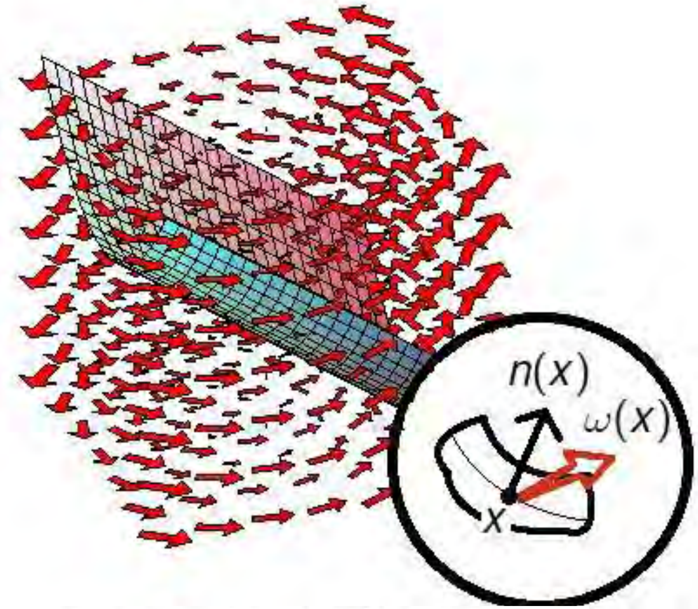


[Glaunes2004] Glaunes, J., Trouve, A., Younes, L. Diffeomorphic matching of distributions: a new approach, … CVPR 2004.

[Durrleman2008] S. Durrleman, X. Pennec, A. Trouvé, P. Thompson, N. Ayache, Inferring Brain Variability from Diffeomorphic Deformations of Currents: an integrative approach, Medical Image Analysis 2008



$$F(\omega) = \sum_i \int_{F_i} \omega(x)^t \tau_i(x) dx$$

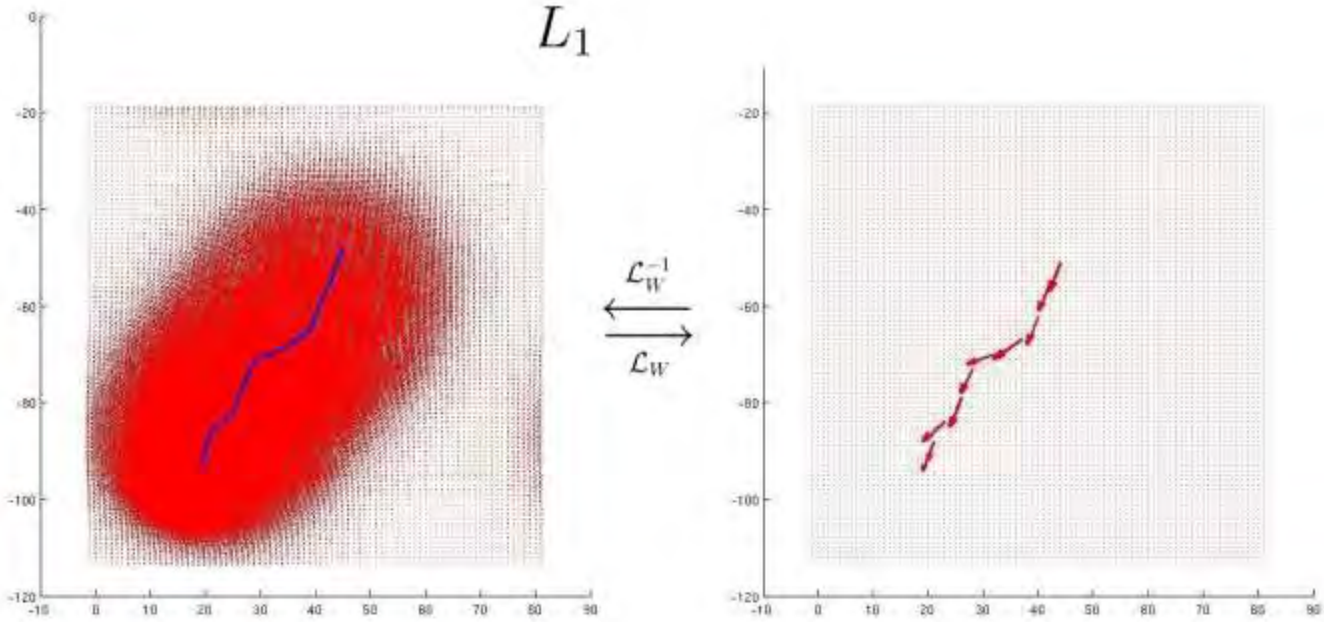


$$S(\omega) = \int_S \omega(x)^t n(x) d\sigma(x)$$

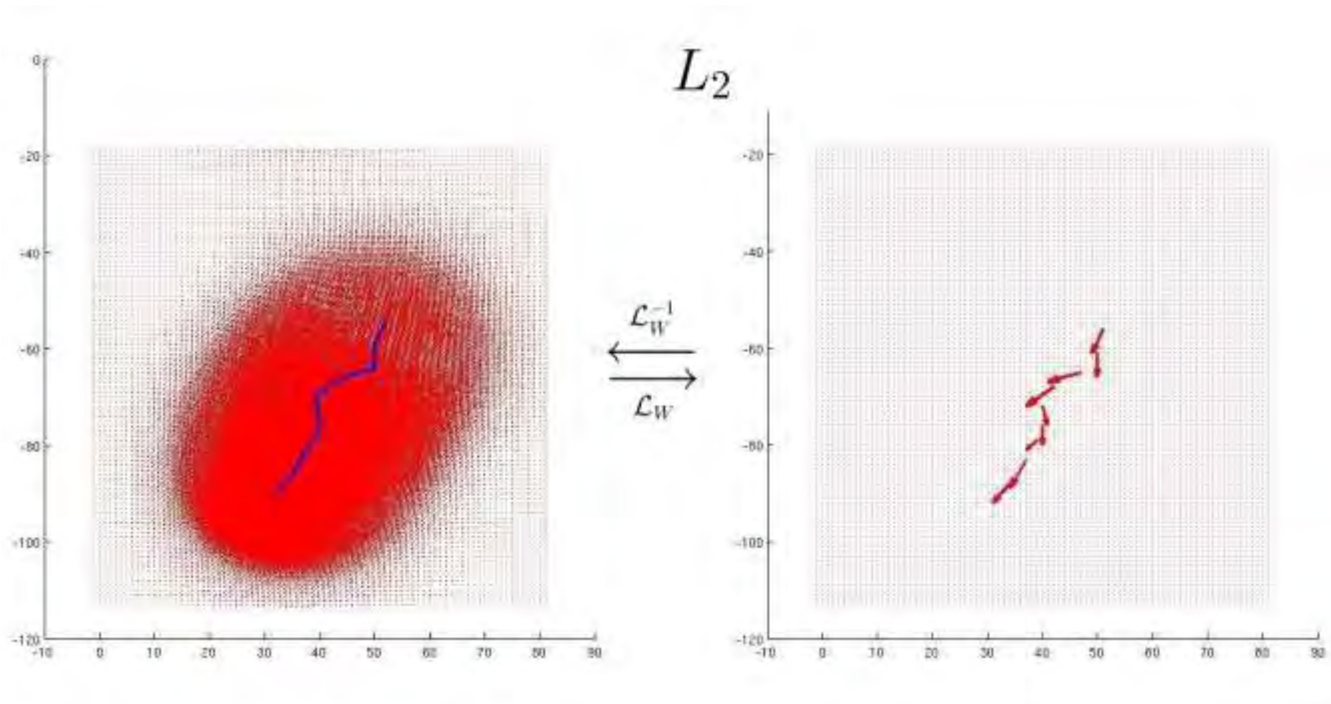
Currents integrate vector field:

- W: test space of vector fields (Hilbert space)
- W*: the space of continuous maps W->R
- W* includes smooth curves, polygonal lines, surfaces, meshes.

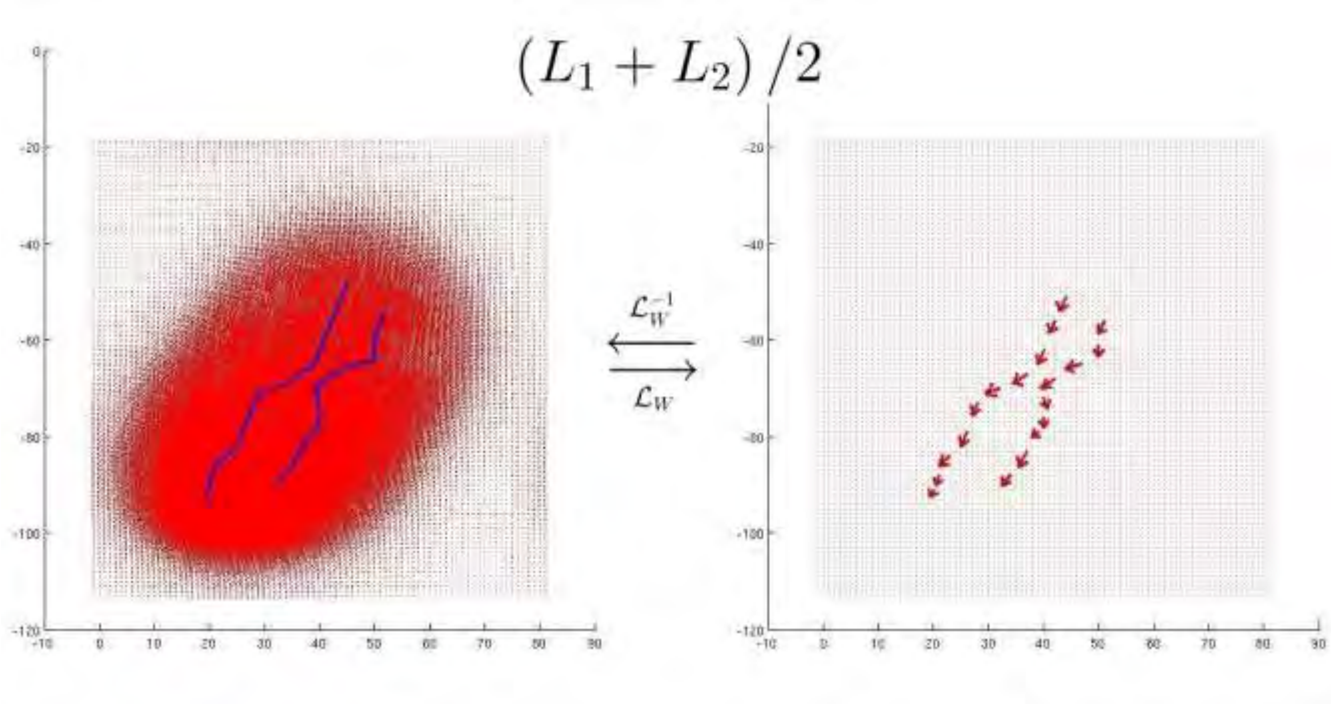
[Vaillant and Glaunès IPMI'05, Glaunès PhD'06]



Durrleman, PhD thesis, 2010

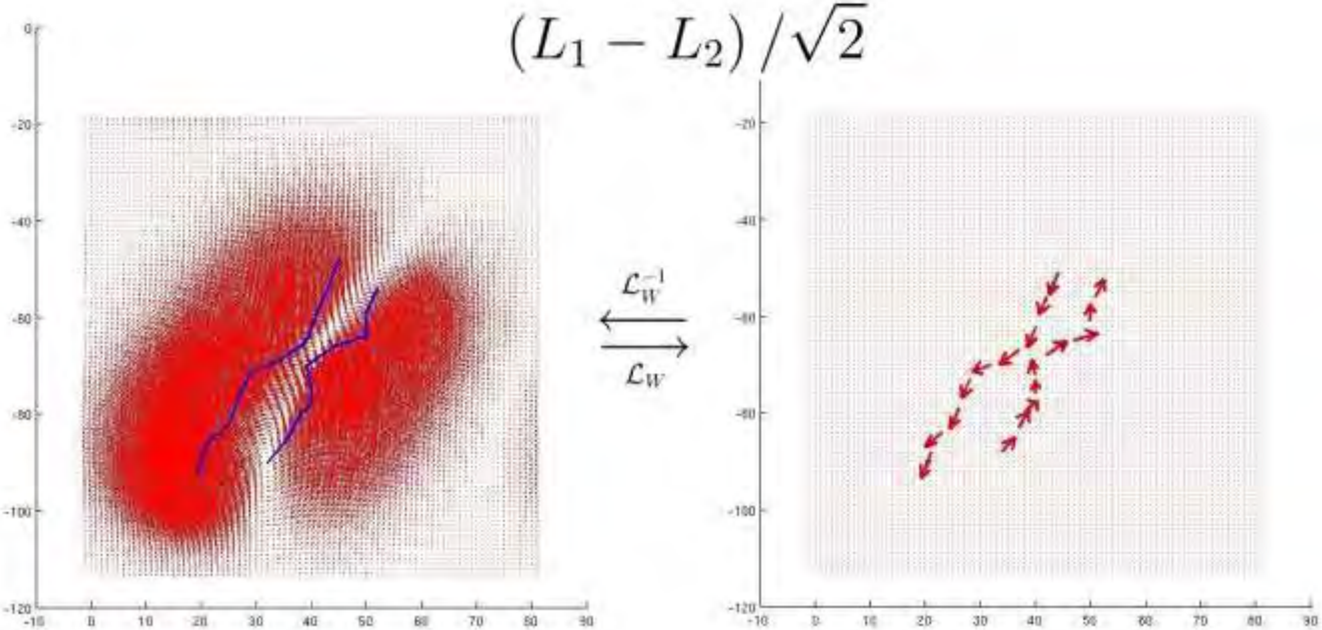


Durrleman, PhD thesis, 2010



Durrleman, PhD thesis, 2010

$$\left(L_1 - L_2\right)/\sqrt{2}$$



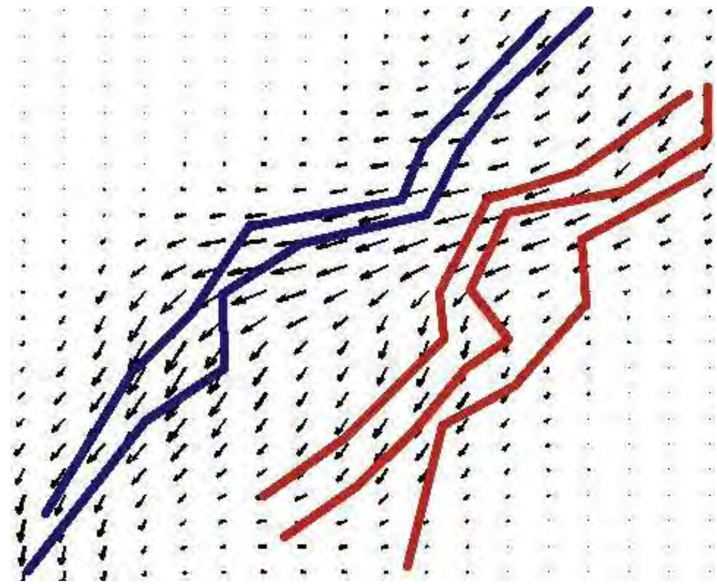
Durrleman, PhD thesis, 2010

The space of currents: a vector space

- Addition = union
- Scaling = weighting different structures
- Sign = orientation

Distance between shapes:

- No point correspondence
- No individual line correspondence
- Robust to line interruption
- Need consistent orientation of lines/surfaces
- Is a norm

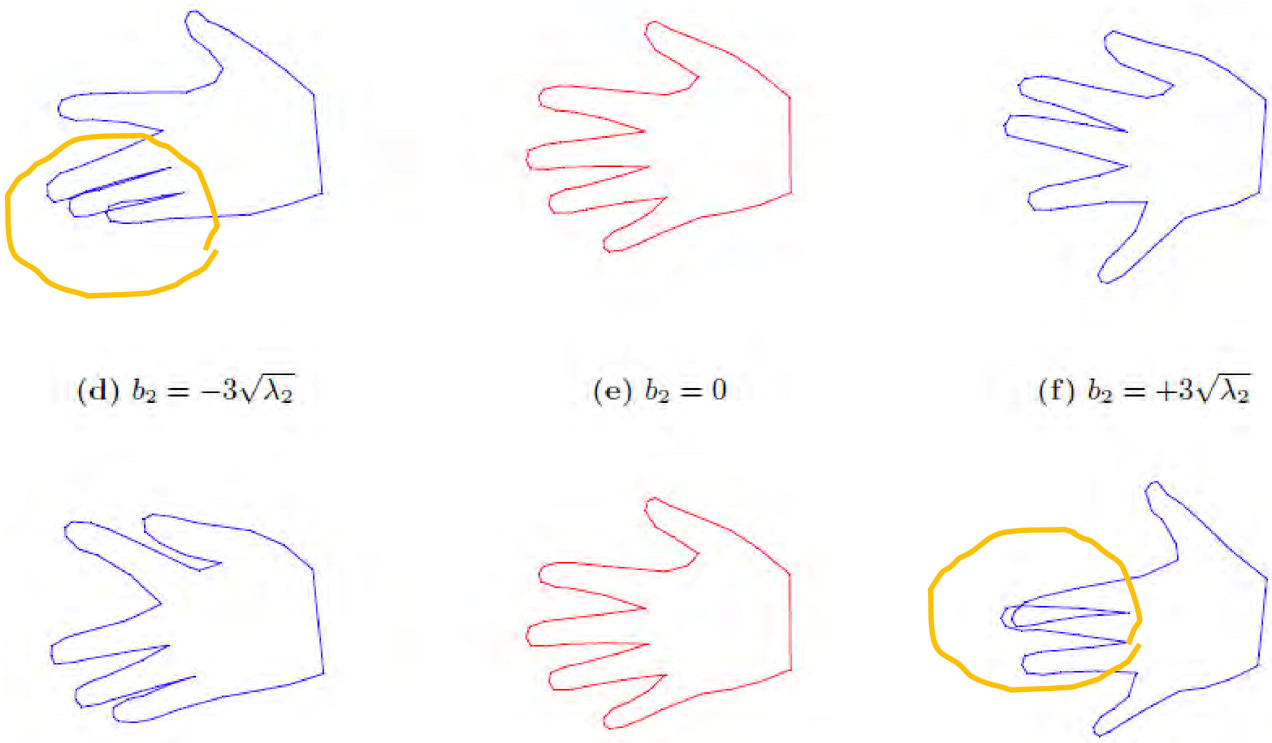


Durrleman, PhD thesis, 2010, JCV 2013

Limitations of Kendall Shape Space

- Shape Space depends on correspondence & parametrization.
- Correspondence still an issue, not defined for shapes with varying topology/resolution etc.
- Statistics on “precise” high-dim (often oversampled) descriptions of shape rather than deformations.
- (PCA Problem: Cannot handle cases well outside of the training set (e.g. occlusions, highly deformable objects).)

Critical Assessment

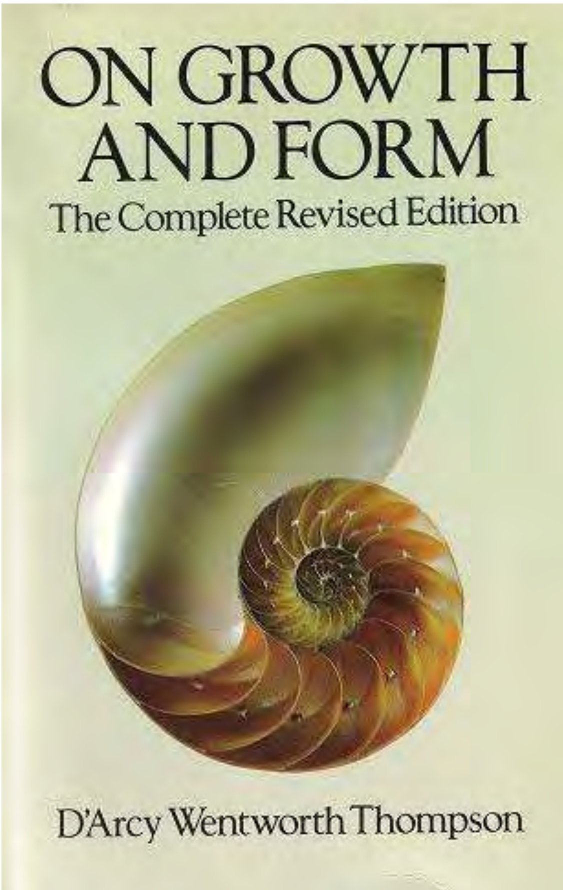


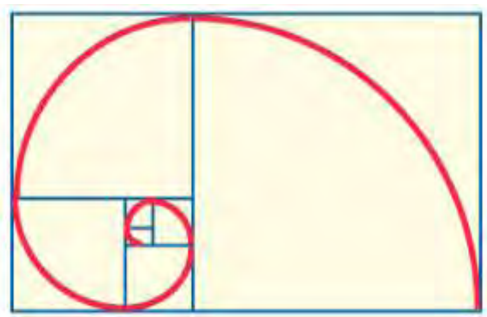
(d) $b_2 = -3\sqrt{\lambda_2}$ (e) $b_2 = 0$ (f) $b_2 = +3\sqrt{\lambda_2}$

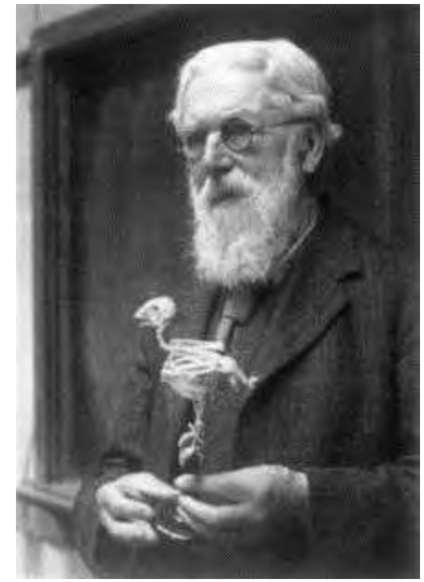
(g) $b_3 = -3\sqrt{\lambda_3}$ (h) $b_3 = 0$ (i) $b_3 = +3\sqrt{\lambda_3}$

Stegmann and Gomez, 2002

Highly Recommended Reading







http://archive.org/download/ongrowthform00thom/ongrowthform00thom.pdf
http://ia700301.us.archive.org/10/items/ongrowthform00thom/ongrowthform00thom.pdf

D'Arcy Wentworth Thompson, On Growth and Form (1917, mathematics and biology)

Shape Spaces: Kendall vs. Deformations

Kendall Shape Space:

- We are interested in the way the points of a shape move (or displace), but there is no general concept of a deformation -- analysis is based on the parameterization of the shape.
- Shape Space forms a complex projective space $\mathbb{CP}^{k-2}$.

Shape Spaces: Kendall vs. Deformations

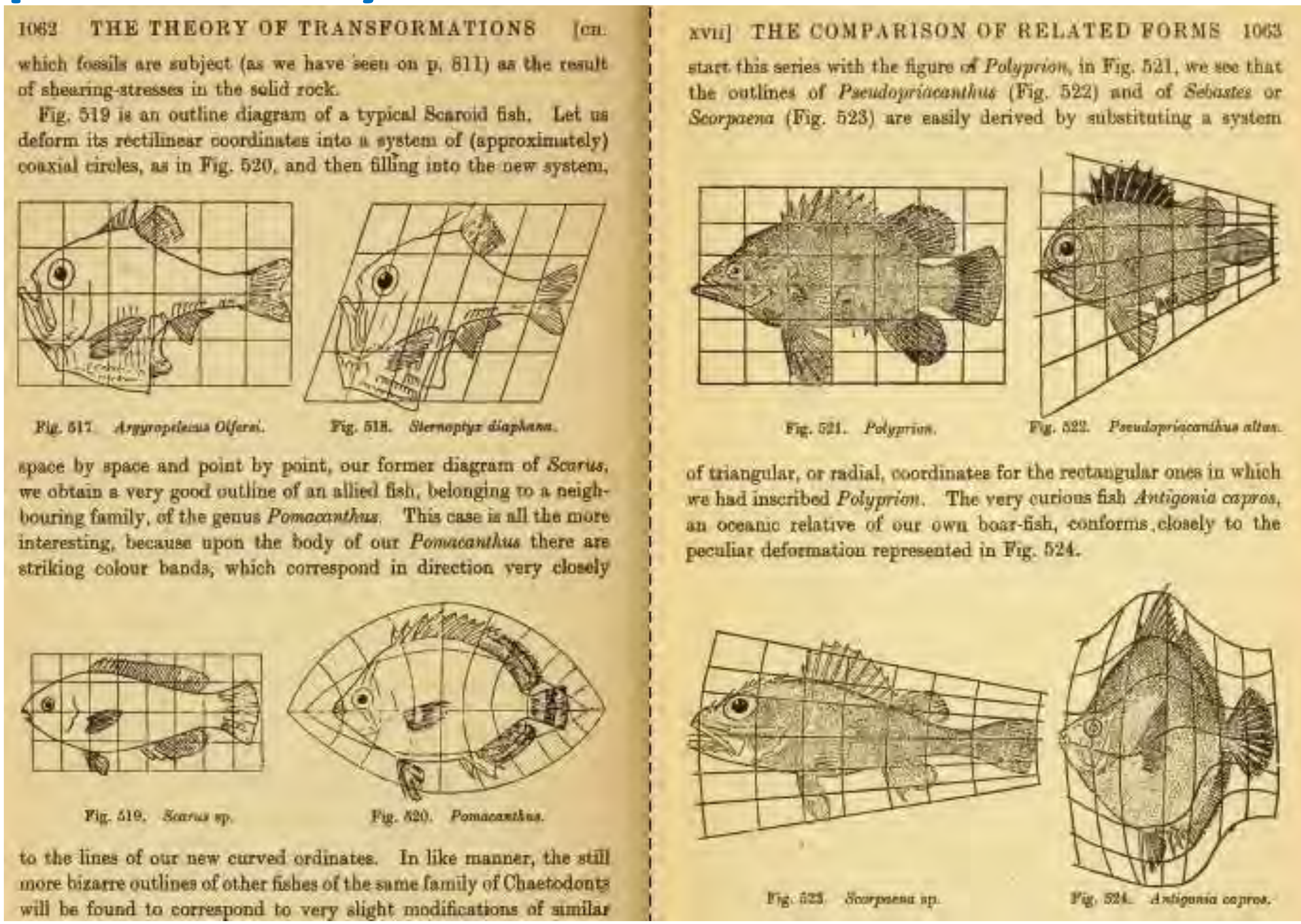
Kendall Shape Space:

- We are interested in the way the points of a shape move (or displace), but there is no general concept of a deformation -- analysis is based on the parameterization of the shape.
- Shape Space forms a complex projective space $\mathbb{CP}^{k-2}$.

D'Arcy Thompson inspired deformation based analysis:

- Interested in the way the ambient space deforms.
- Statistical analysis is centered on the deformations of space, not movement & displacement of points on shapes.
- What is the Shape Space? Information via deformations.

Shape Analysis via Transformations


1062 THE THEORY OF TRANSFORMATIONS [CH.

which fossils are subject (as we have seen on p. 811) as the result of shearing-stresses in the solid rock.

Fig. 519 is an outline diagram of a typical Scaroid fish. Let us deform its rectilinear coordinates into a system of (approximately) coaxial circles, as in Fig. 520, and then filling into the new system,

Fig. 517. *Argyropelecus Olfersi.* Fig. 518. *Sternoptyx diaphana.*

space by space and point by point, our former diagram of *Scarus*, we obtain a very good outline of an allied fish, belonging to a neighbouring family, of the genus *Pomacanthus*. This case is all the more interesting, because upon the body of our *Pomacanthus* there are striking colour bands, which correspond in direction very closely

Fig. 519. *Scarus* sp. Fig. 520. *Pomacanthus.*

to the lines of our new curved ordinates. In like manner, the still more bizarre outlines of other fishes of the same family of Chaetodonts will be found to correspond to very slight modifications of similar

XVII] THE COMPARISON OF RELATED FORMS 1063

start this series with the figure of *Polyprion*, in Fig. 521, we see that the outlines of *Pseudopriacanthus* (Fig. 522) and of *Sebastes* or *Scorpaena* (Fig. 523) are easily derived by substituting a system

Fig. 521. *Polyprion.* Fig. 522. *Pseudopriacanthus altus.*

of triangular, or radial, coordinates for the rectangular ones in which we had inscribed *Polyprion*. The very curious fish *Antigonia capros*, an oceanic relative of our own boar-fish, conforms closely to the peculiar deformation represented in Fig. 524.

Fig. 523. *Scorpaena* sp. Fig. 524. *Antigonia capros.*

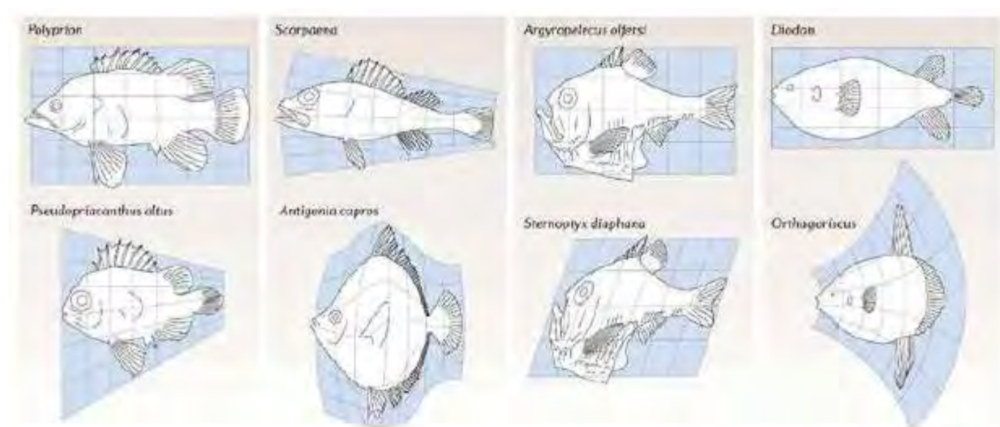
D'Arcy Thompson introduced the Method of Coordinates to accomplish the process of comparison.

D'Arcy Wentworth Thompson, On Growth and Form (1917, mathematics and biology)

Biological variation through mathematical transforms

D'Arcy Thompson laid out his vision in his treatise “On Growth and Form“. In 1917 he wrote:

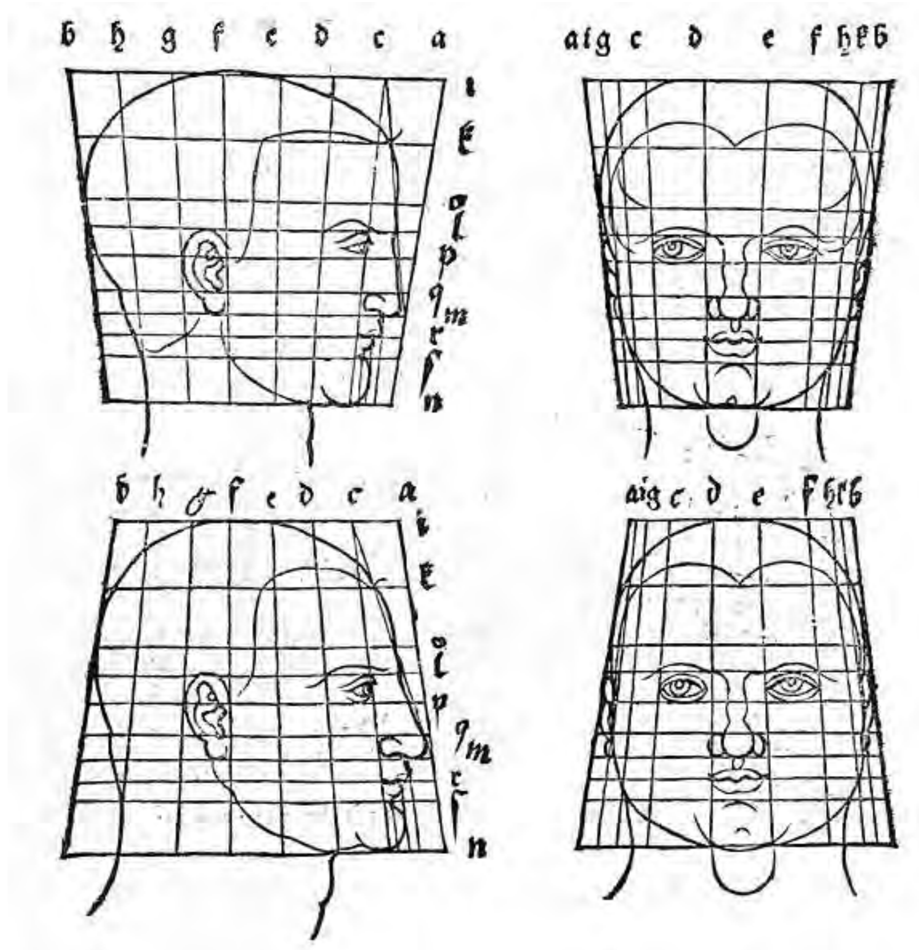
In a very large part of morphology, our essential task lies in the comparison of related forms rather than in the precise definition of each; and the deformation of a complicated figure may be a phenomenon easy of comprehension, though the figure itself may be left unanalyzed and undefined."



Even earlier…

Albrecht Dürer (1471-1528): German painter, printmaker, engraver and mathematician.

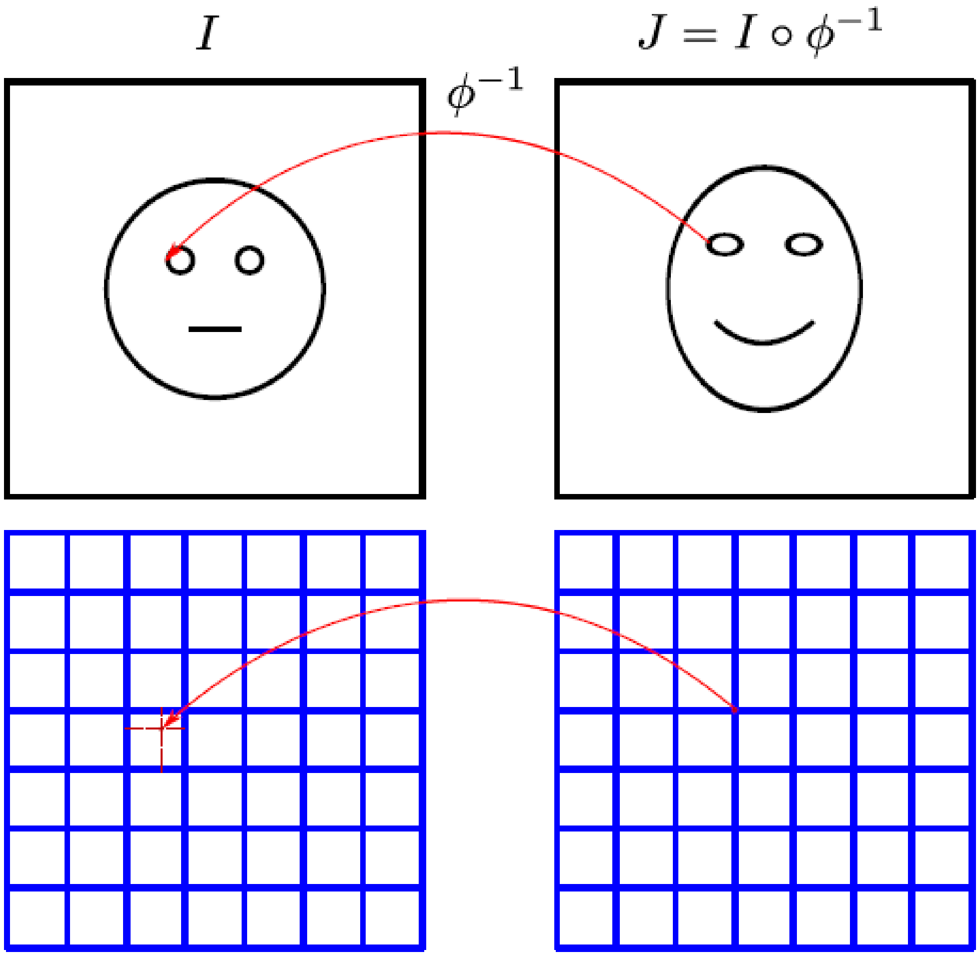
Studies of human proportions.
http://commons.wikimedia.org/wiki/Albrecht_Durer



Face transformations by Albrecht Dürer

http://commons.wikimedia.org/wiki/File:Durer_face_transforms.jpg

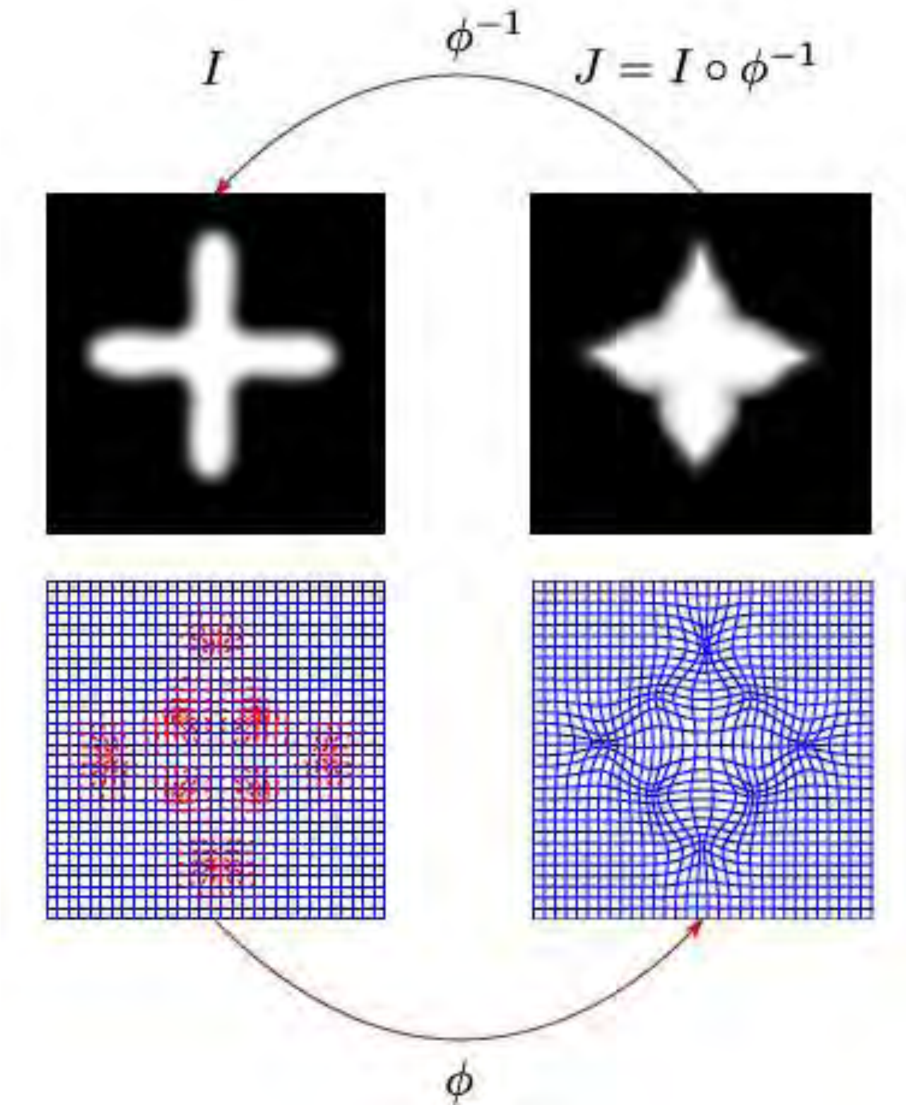
Ambient Space Deformation



Change in geometric entities in images represented as transformations of the underlying coordinate grid.

Nikhil Singh, PhD thesis Utah 2013

Ambient Space Deformation



Initial velocity* as a smooth vector field and the corresponding diffeomorphic flow that transforms the shape “plus" to “flower".

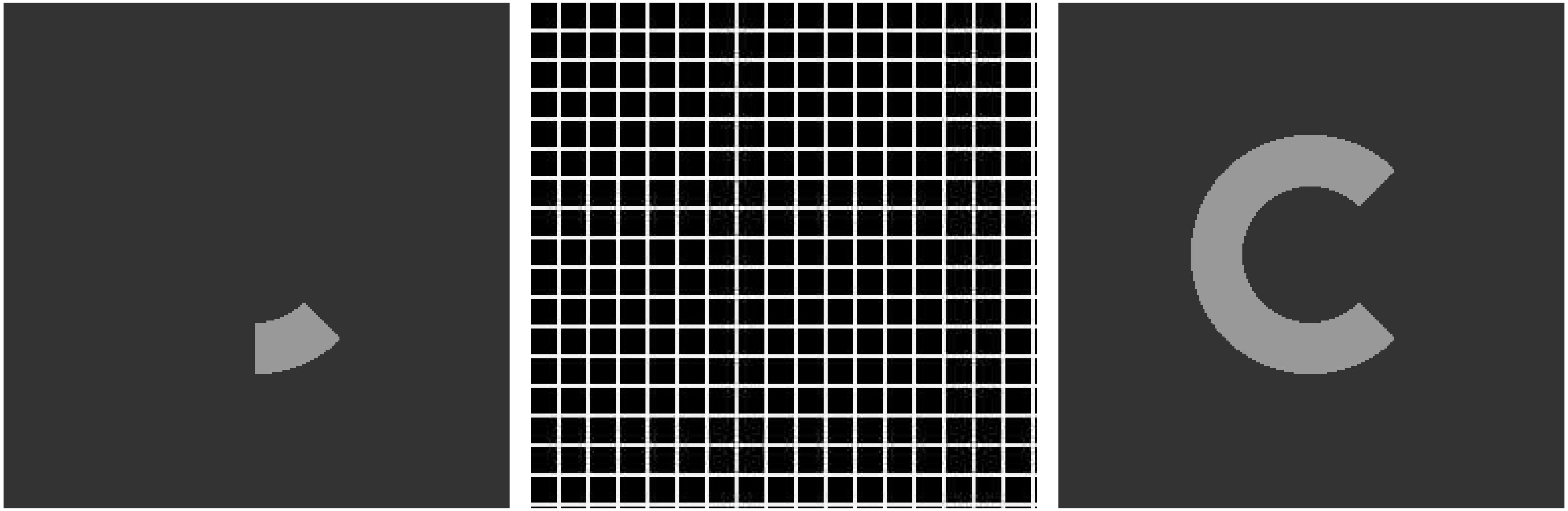
*velocity: momenta after convolution with kernel

Nikhil Singh, PhD thesis Utah 2013

Concept of Diffeomorphism

Diffeomorphisms:

- one-to-one onto (invertible) and differential transformations
- preserves topology



Slide courtesy Sarang Joshi

Large Deformation Diffeomorphic Metric Mapping (LDDMM)

- Space of all Diffeomorphisms $Diff\,(\Omega)$ forms a group under composition:

$$\forall h_1, h_2 \in Diff\,(\Omega) : h = h_1 \circ h_2 \in Diff\,(\Omega)$$

- Space of diffeomorphisms <u>not a vector space</u>.

$$\forall h_1, h_2 \in Diff\,(\Omega) : h = h_1 + h_2 \notin Diff\,(\Omega)$$

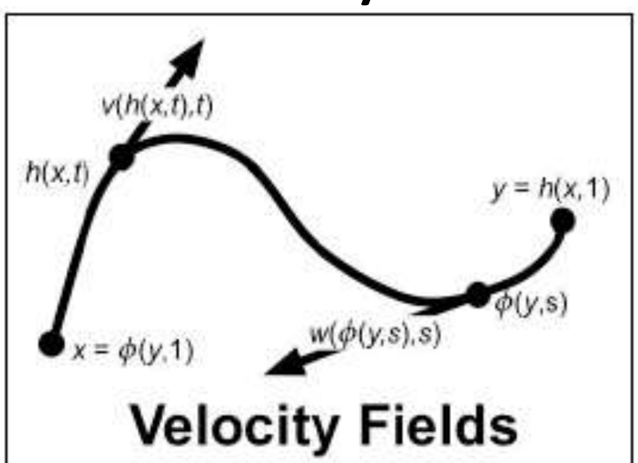
- Small deformations, or “Linear Elastic” registration approaches, ignore these two properties.

Slide courtesy Sarang Joshi

Large deformation diffeomorphisms.

- $Diff(\Omega)$ infinite dimensional "Lie Group".
- Tangent space: The space of smooth vector valued velocity fields on Ω .
- Construct deformations by integrating flows of velocity fields.
- Induce a metric via a differential norm on velocity fields.

$$\frac{d}{dt} h(x;t) = v(h(x;t);t) \quad h(x;0) = x:$$



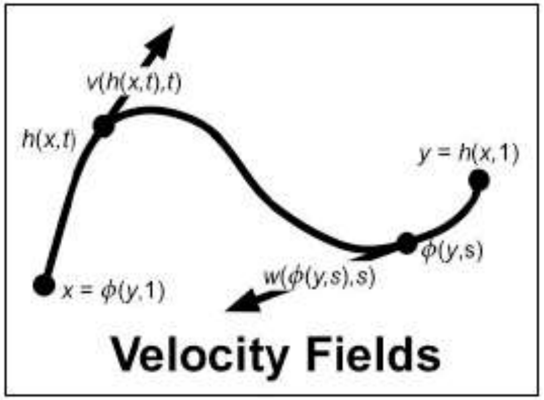
$$y = h(x,1) = x + \int_0^1 v(h(x,\tau),\tau)d\tau$$

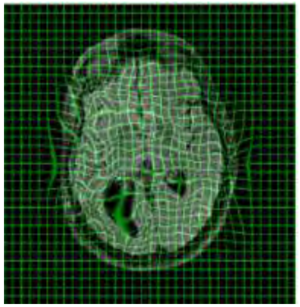
$$x = \phi(y,1) = y + \int_0^1 w(\phi(y,\tau),\tau)d\tau$$

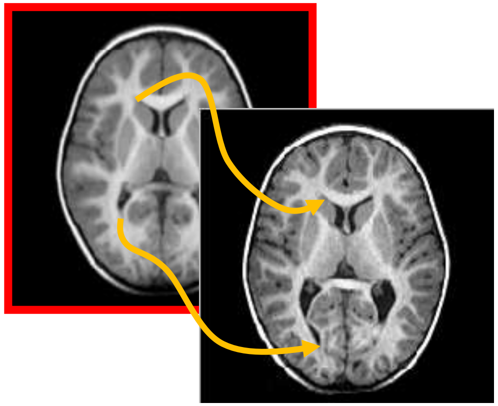
Construction of Diffeomorphisms

Diffeomorphisms:

- Construct deformations by integrating flows of velocity fields.
- Induce a metric via a differential norm on velocity fields.
- Distance btw. two diffeomorphisms: $D(h_1, h_2) = D\left(e, {h_1}^{-1} \circ h_2\right)$ (metric).

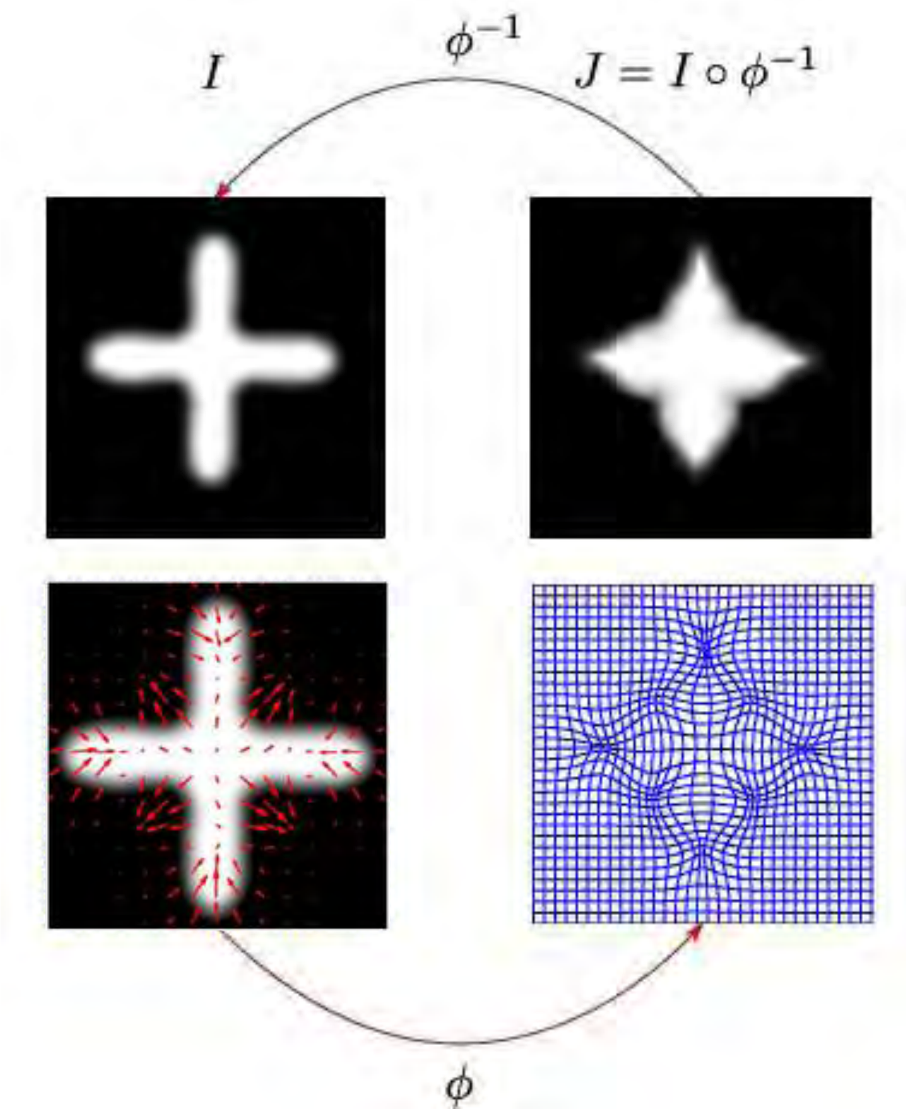






Miller, Christensen, Joshi / Joshi et al., Neuroimage 2004

Ambient Space Deformation



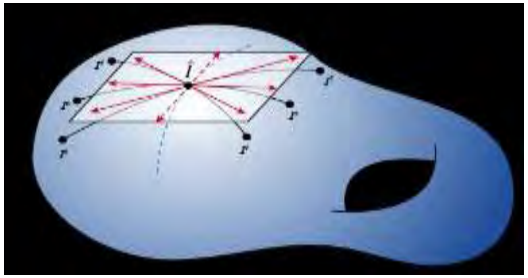
Momenta and the corresponding diffeomorphic flow that transforms the shape “plus" to “flower".

Nikhil Singh, PhD thesis Utah 2013
James Fishbaugh, GSI conference, Springer 20014

Momenta and Statistics

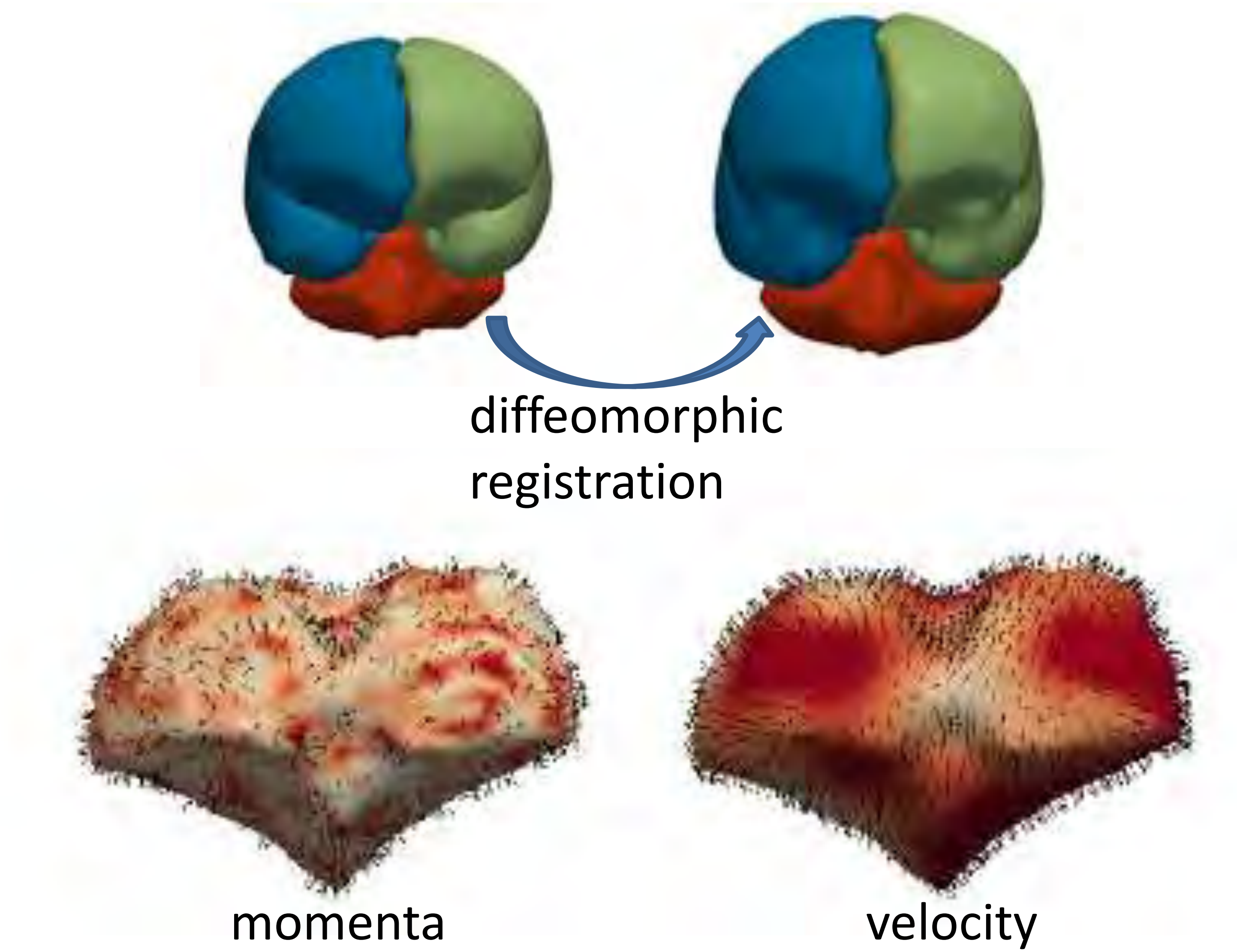
- The momenta field plays the role of the tangent vector in the Riemannian sense → **Momenta exist in a linear space.**
- Analysis of geometrical variability: PCA on the feature vectors of deformations → PCA* by computing mean and covariance matrix of momenta.

(*kernel PCA for currents)



Miller, Trouve, Younes, On the metrics and euler-lagrange equations of computational anatomy. Annu Rev Biomed Eng.2002
Durrleman et al., NeuroImage 2010

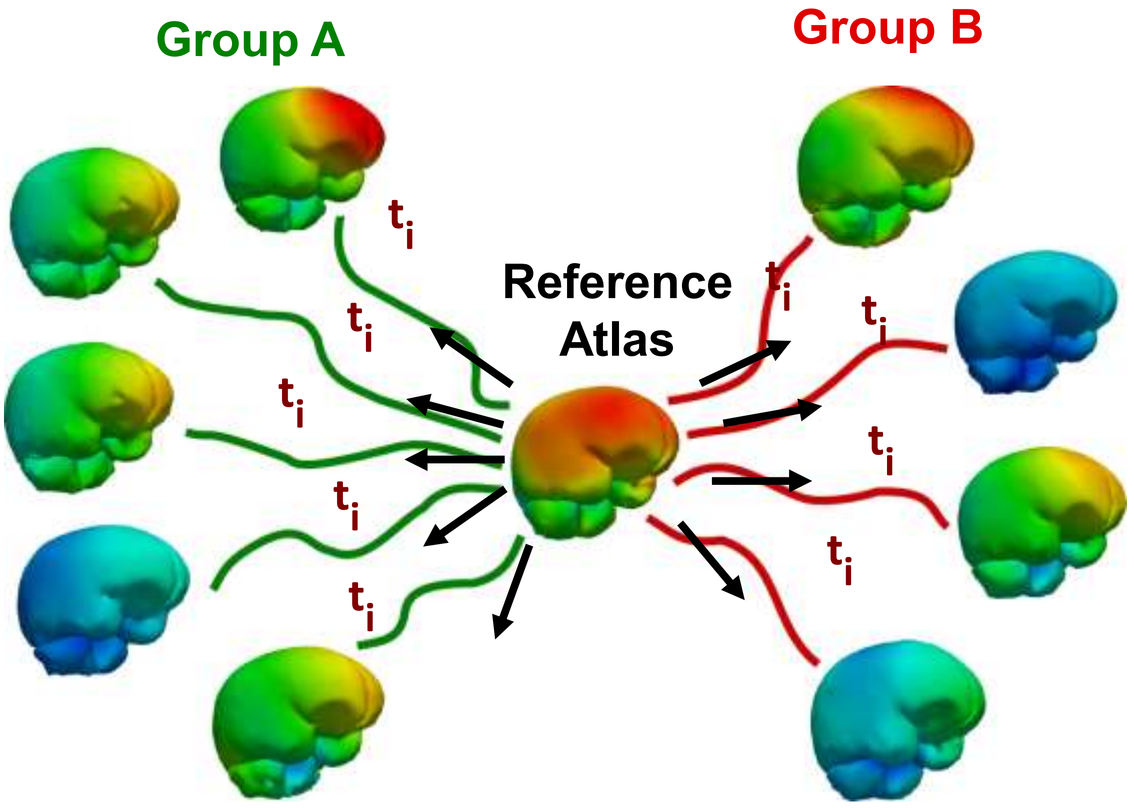
Geodesic Flow: Initial Momenta



Fishbaugh, Durrleman, Gerig, MICCAI 2011, 2012, 2013

Statistics on Deformations

Flows of diffeomorphisms are **geodesic** → initial momenta parameterize deformation.

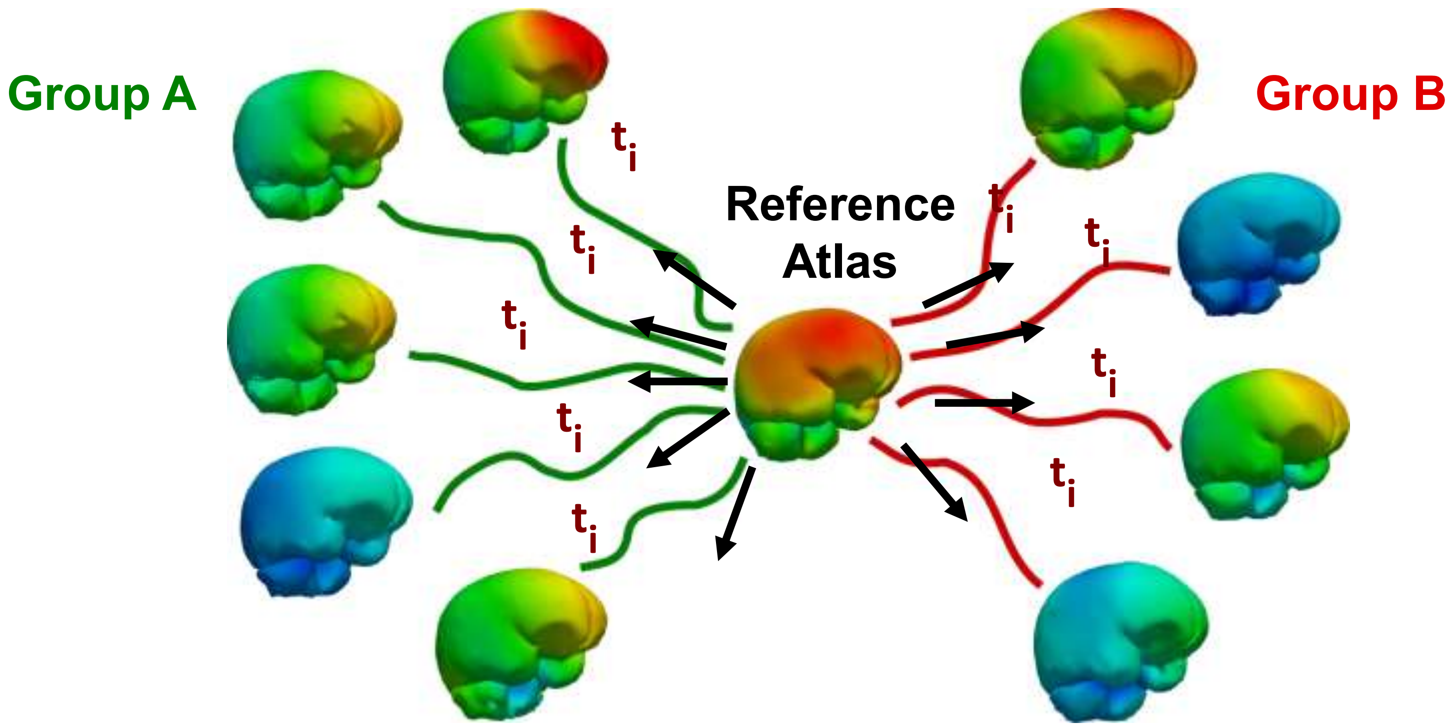


Fishbaugh, Durrleman, Gerig, MICCAI 2012

Statistics on Deformations

Flows of diffeomorphisms are **geodesic** → initial momenta parameterize deformation.

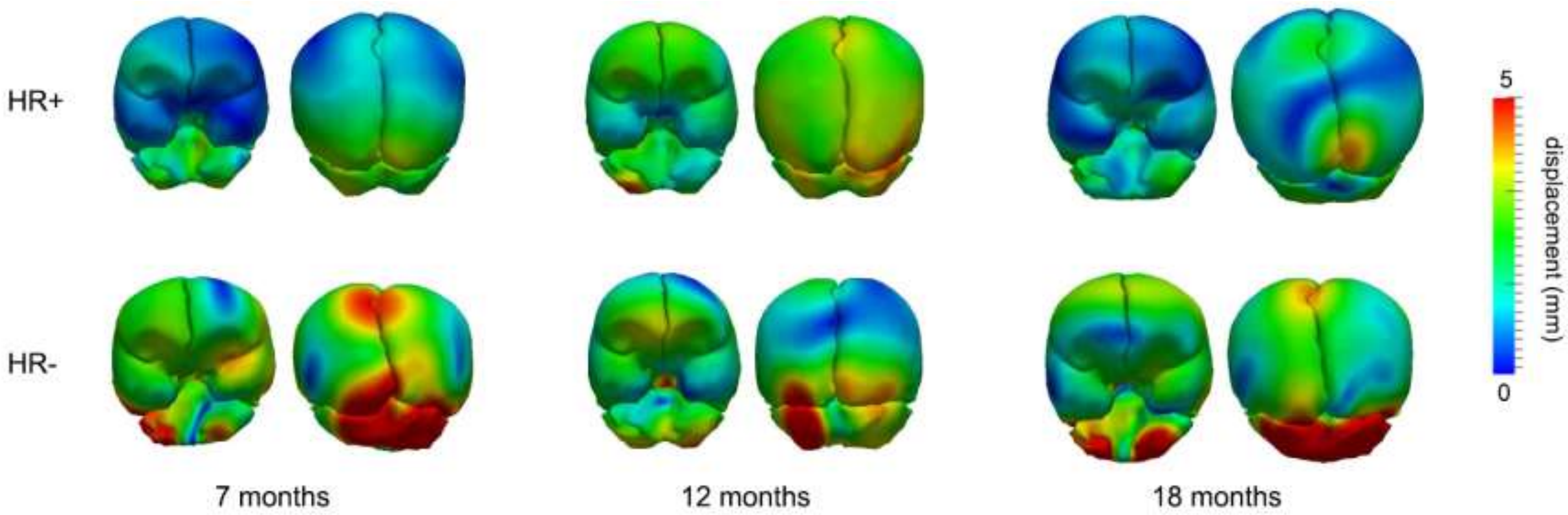
Geodesics from atlas to each subject share the same tangent space, so we can perform linear operations on the momenta, such as computing the mean and variance.



Fishbaugh, Durrleman, Gerig, MICCAI 2012

Clinical Application: Autism

First mode of deformation from **PCA** per age group, explaining the variability of each group w.r.t. the normative reference atlas.



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Hypothesis testing → **no significant** differences in magnitude of initial momenta

Momenta and Statistics

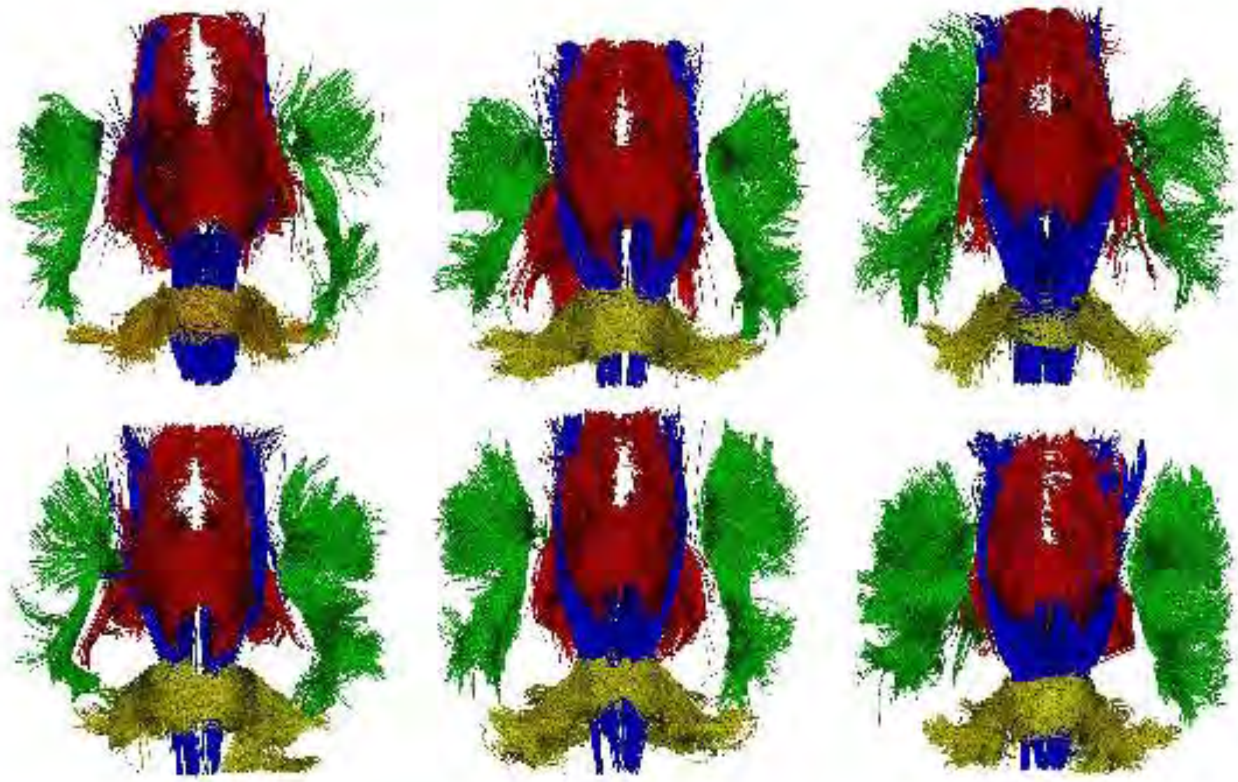


Figure 8: Five fiber bundles extracted in six subjects using MedINRIA. **Blue:** the corticospinal tract. **Yellow:** the corticobulbar tract. **Red:** the callosal fibers. **Green:** the left and right arcuate fasciculi.

Durrleman et al., NeuroImage 2010

Momenta and Statistics

(a) One subject (b) template (occipital view) (c) template (lateral view)

Figure 12: **Template of five bundles**: the corticospinal tract (blue), the corticobulbar tract (yellow), the callosal fibers (red), the left and right arcuate fasciculi (green). (a): one subject among the six of the data set. (b,c) the atlas estimated such that original data result from random deformations of the template plus random perturbations.

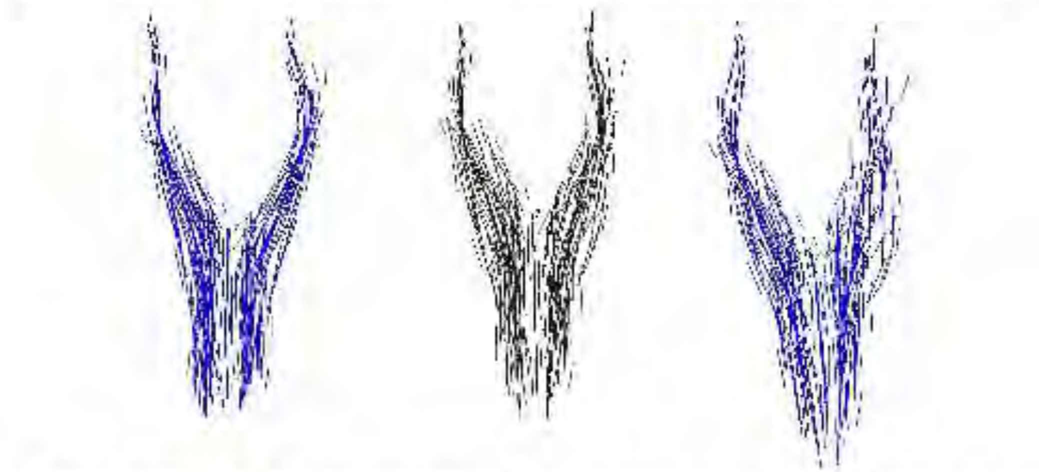
def. mode at $-\sigma$ template def. mode at $+\sigma$

a- **First mode of deformation at** $\pm\sigma$

Durrleman et al., NeuroImage 2010

Momenta and Statistics

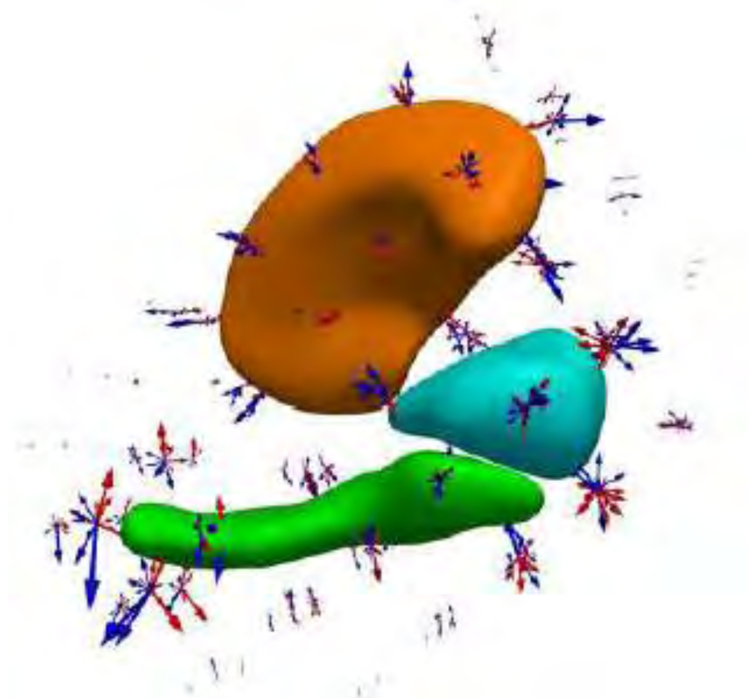
a- **1st mode of deformation** of the cortico-spinal tract (lateral view)



b- **2nd mode of deformation** of the cortico-spinal tract (frontal view)

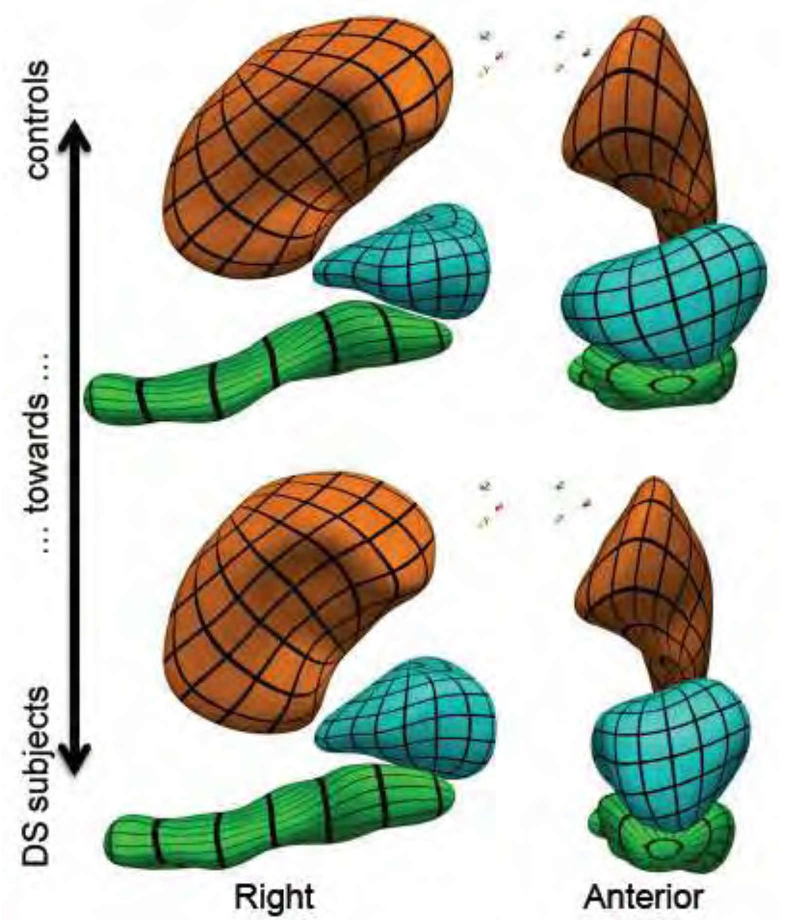
Durrleman et al., NeuroImage 2010

Statistics on Deformations



Geodesics from atlas to each subject share the same tangent space.

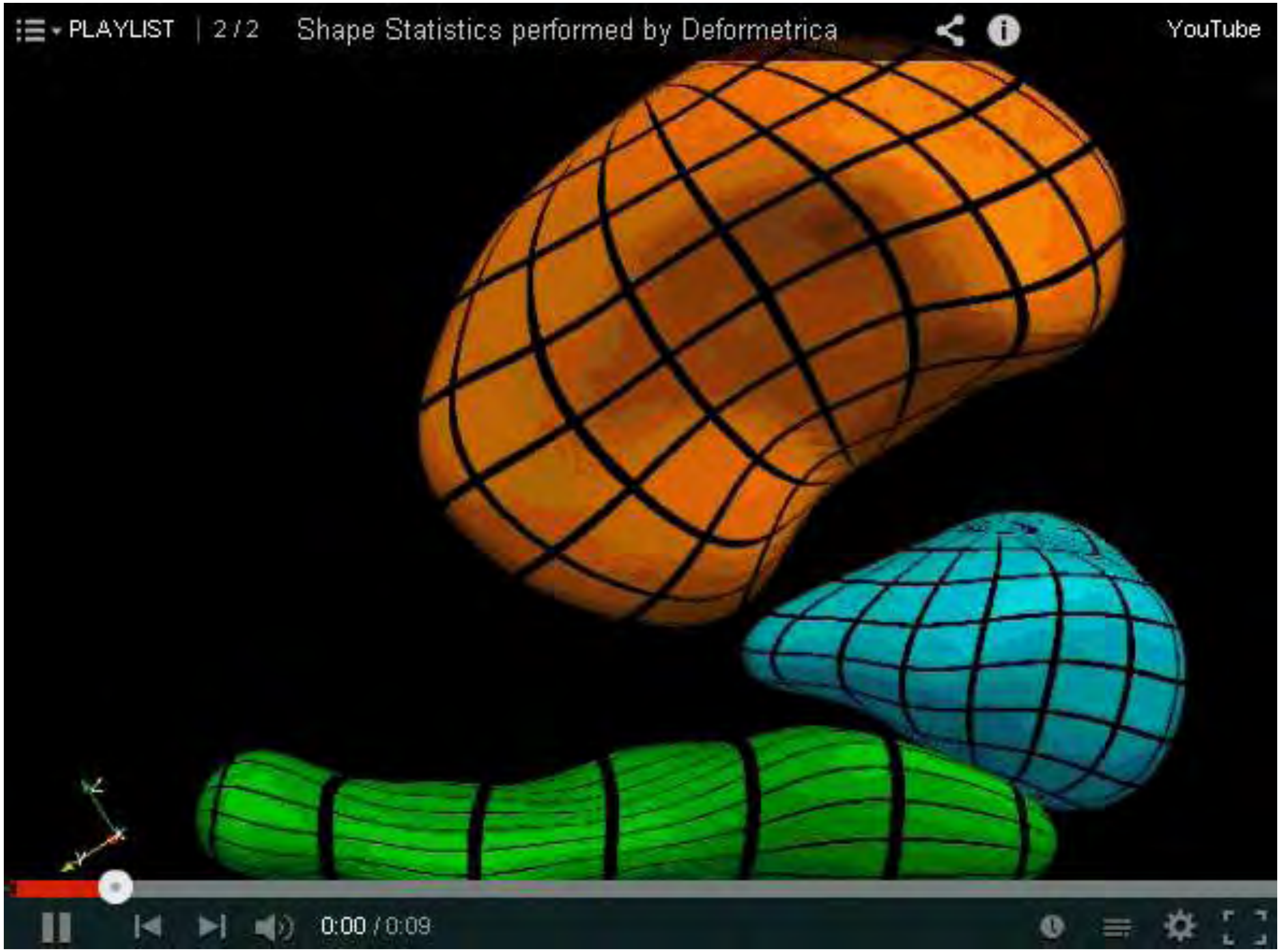
Momentum vectors of DS subjects (red) and controls (blue) in atlas coordinate space.



Most discriminative deformation axis between Down's and Controls.

Durrleman et al, Neuroimage 2014

Statistics on Deformations

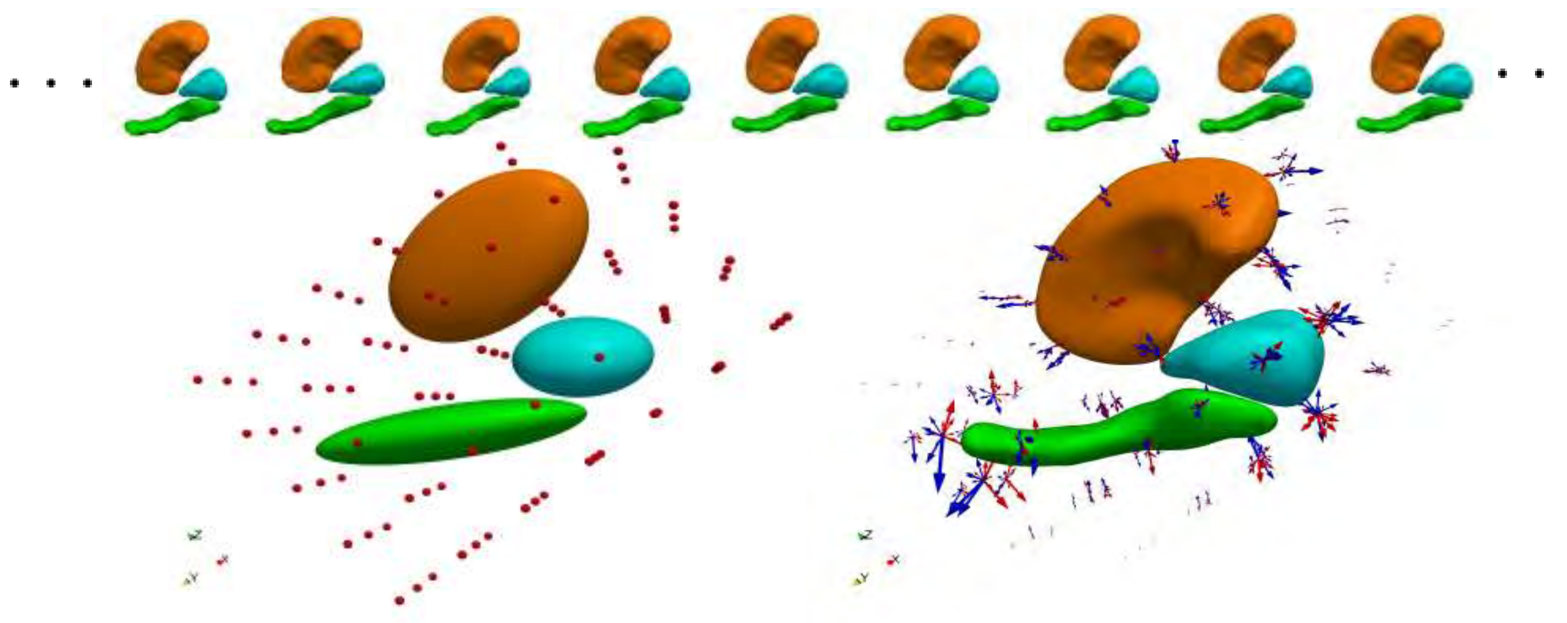


Most discriminative deformation axis between Down's Syndrome and Controls.

Durrleman et al, Neuroimage 2014

Deformetrics with Sparsity:

Tackling fundamental problem of high-dim features & low-dim sample size (HDLSS)

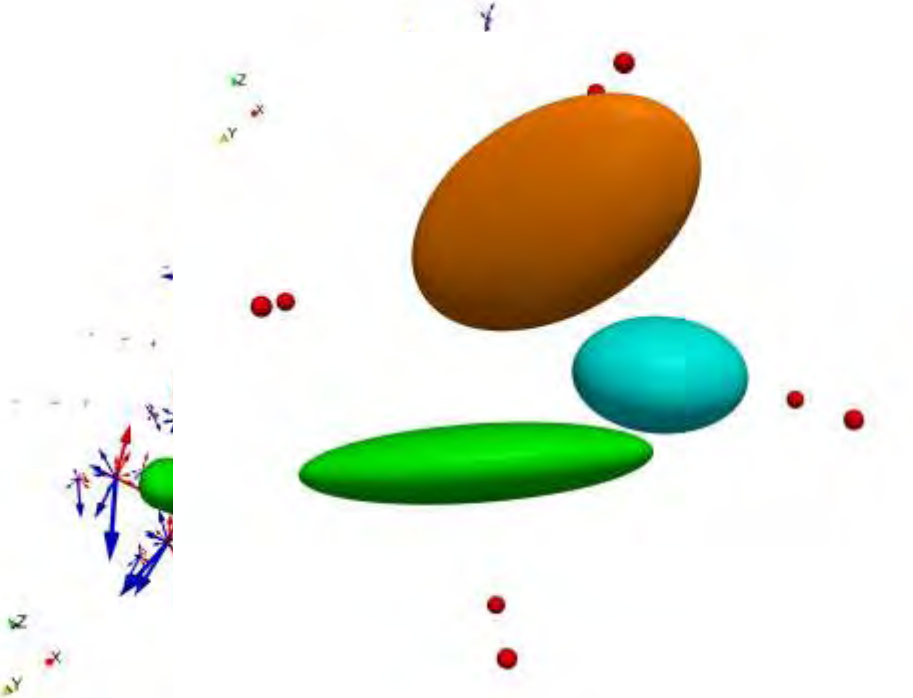


Initialization Atlas

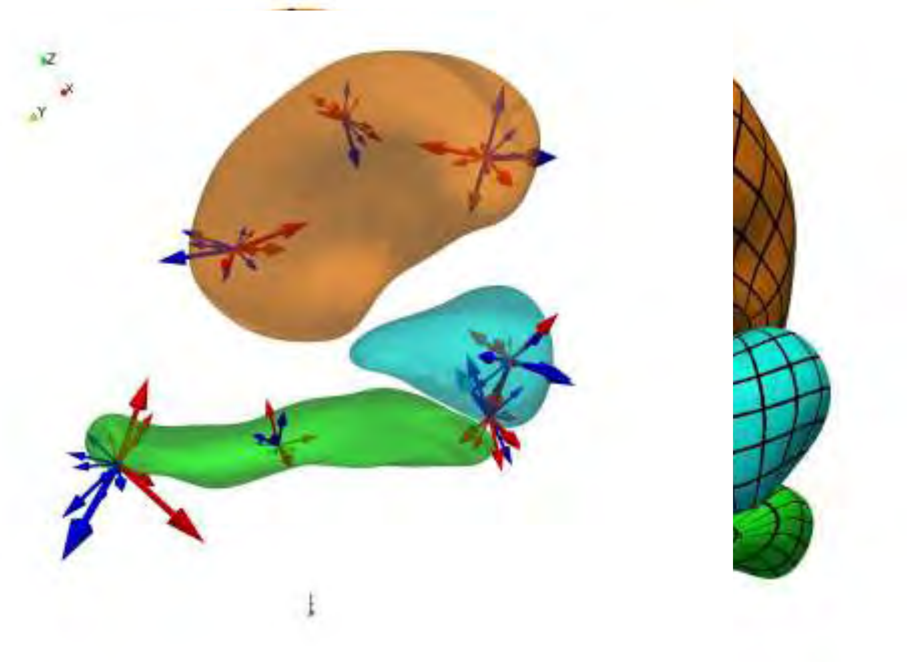
Image evolution described by considerably **fewer** parameters, **Concentrated** in areas undergoing most dynamic changes

Durrleman, 2013 / Fishbaugh IPMI 2013, GSI 2013

Statistics: Ambient Space Deform.



Common template for
8 Down's syndrome patients + 8 Ctrls



Most discriminative axis

→Importance of optimization in control points positions!

Classification (leave-2-out) with 105 control points:

	specificity	sensitivity
Max Likelihood	100% (64/64)	100% (64/64)
LDA	98% (63/64)	100% (64/64)

Classification (leave-2-out) with 8 control points:

	specificity	sensitivity
Max Likelihood	97% (62/64)	100% (64/64)
LDA	94% (60/64)	89% (57/64)

Deformation of Ambient Space

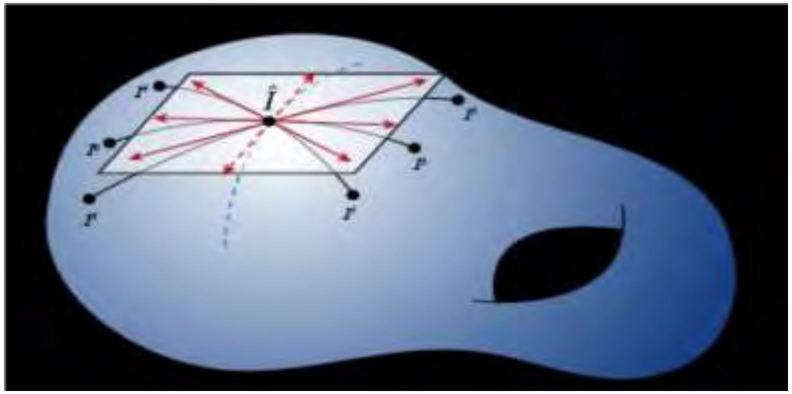
Main advantages:

- Shape space independent on shape representation.
- Natural way to handle multiple shapes, topology variations, combinations of points, lines, contours, image intensity etc.
- Statistics on low #features rather than high-dimensional oversampled shape representation.

Mean and Variability

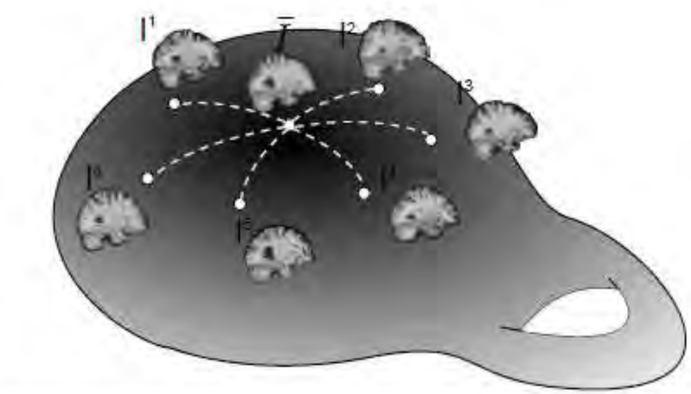
High-dimensional space:

- Variances and covariances
- Non-Euclidean geometry
- Statistics on tangent spaces

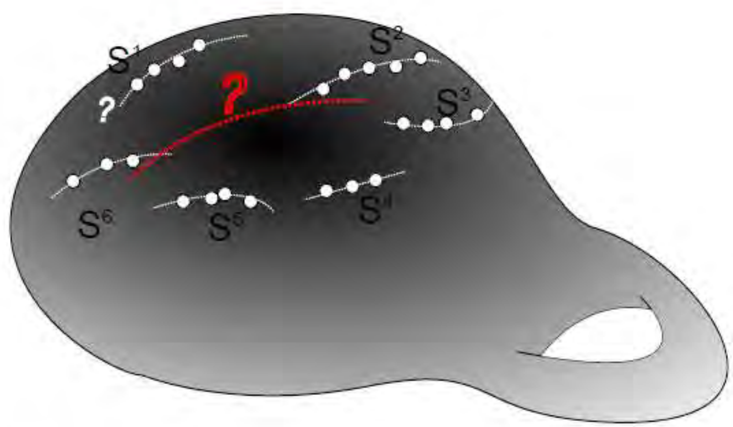


Singh, Fletcher, Joshi et al., ISBI 2013, best paper award

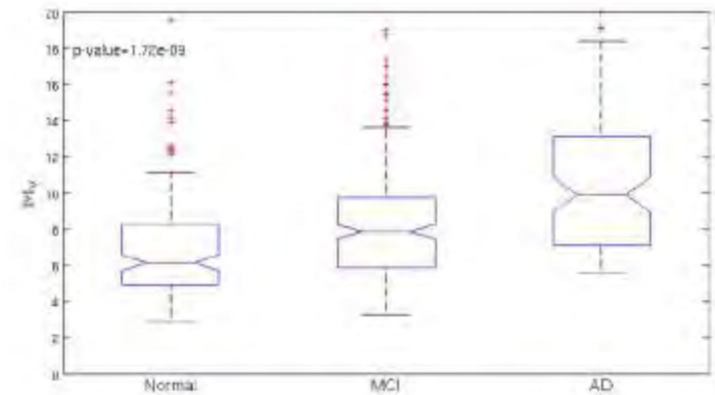
Normative Atlas of 4D Trajectories: Work in Progress



[3] Joshi et. al(2004), Beg et. al(2005)



Repeated scans of anatomy over time and across population.



Group differences in rates of longitudinal atrophy in AD, MCI and Normal control.

Group differences in rates of longitudinal change/atrophy in AD, MCI and Normal control.

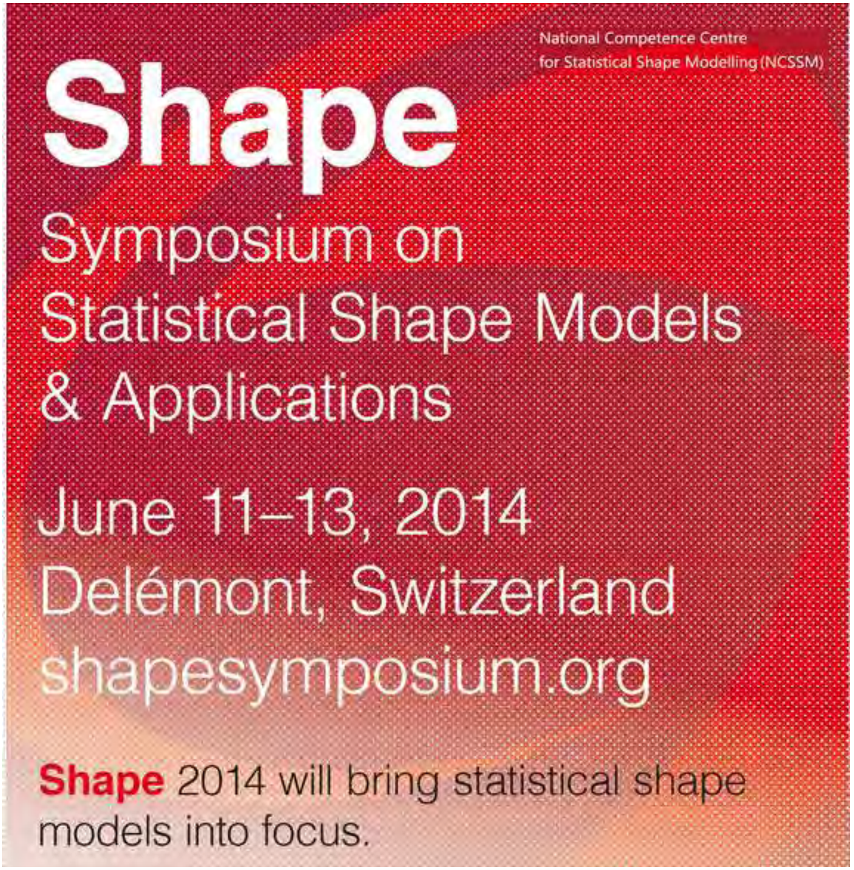
Nikhil Singh et al., ISBI 2013, best paper award

Quotation of the Day

"The perfection of mathematical beauty is such ….. that whatsoever is most beautiful and regular is also found to be the most useful and excellent."

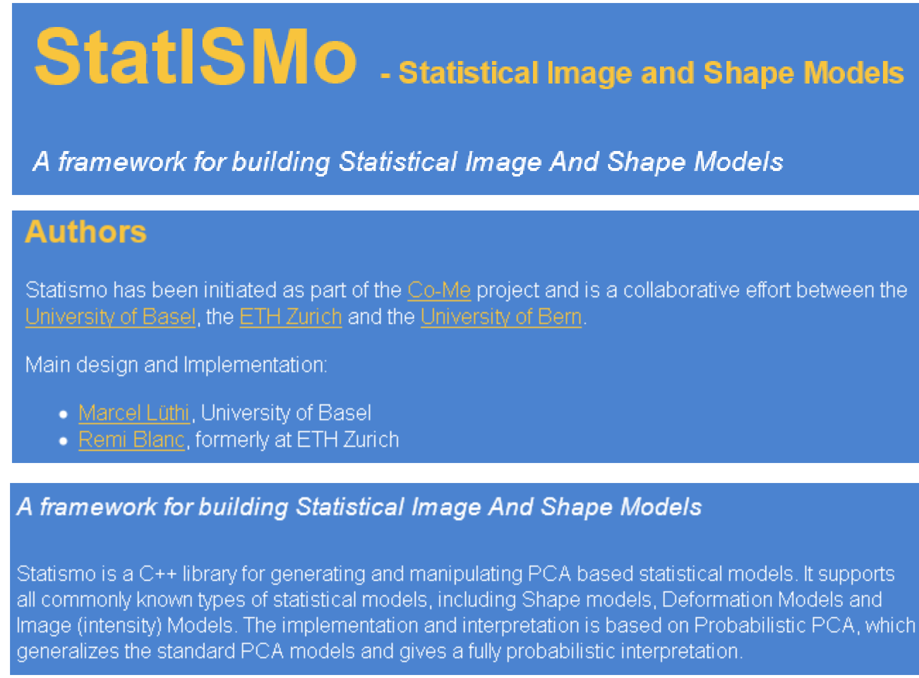
D'Arcy Wentworth Thompson

Software Resources



Keynotes by M. Styner, T. Heimann, Y. Sato, Ch. Lorenz, X. Pennec, G. Gerig

http://www.shapesymposium.org/



StatISMo - Statistical Image and Shape Models

A framework for building Statistical Image And Shape Models

Authors

Statismo has been initiated as part of the Co-Me project and is a collaborative effort between the University of Basel, the ETH Zurich and the University of Bern.

Main design and Implementation:

- Marcel Lüthi, University of Basel
- Remi Blanc, formerly at ETH Zurich

A framework for building Statistical Image And Shape Models

Statismo is a C++ library for generating and manipulating PCA based statistical models. It supports all commonly known types of statistical models, including Shape models, Deformation Models and Image (intensity) Models. The implementation and interpretation is based on Probabilistic PCA, which generalizes the standard PCA models and gives a fully probabilistic interpretation.

http://statismo.github.io/statismo/

Software Resources

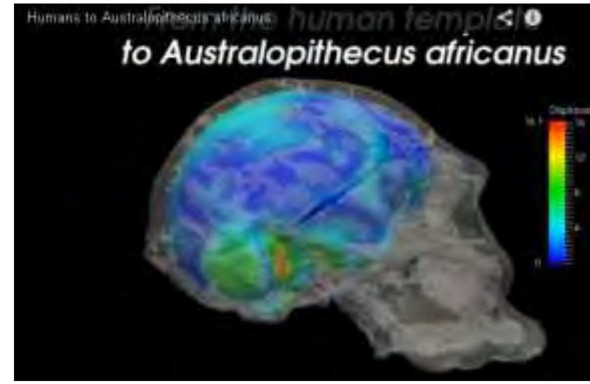
Deformetrica

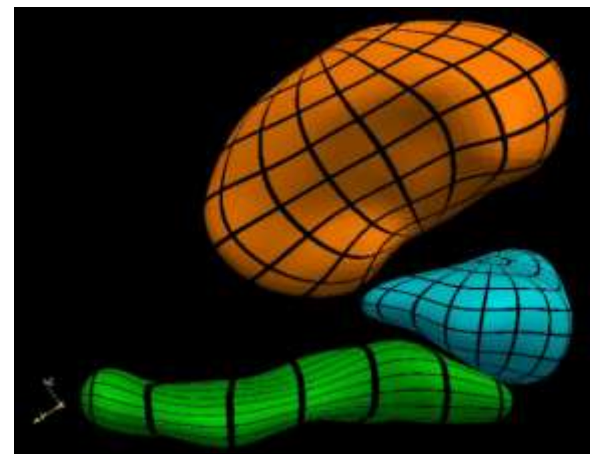
WELCOME! PROJECT DOWNLOADS GET HELP GALLERY FORUM

Contributors:

- Stanley Durrleman (INRIA/ICM Aramis team) *since v1.0*
- Marcel Prastawa (University of Utah - SCI Institute) *for v1.0*
- Alexandre Routier (INRIA/ICM Aramis team funded by CATI) *since v2.0*

Deformetrica is a software for the statistical analysis of 2D and 3D shape data. It essentially computes deformations of the 2D or 3D ambient space, which, in turn, warp any object embedded in this space, whether this object is a curve, a surface, a structured or unstructured set of points, or any combination of them.



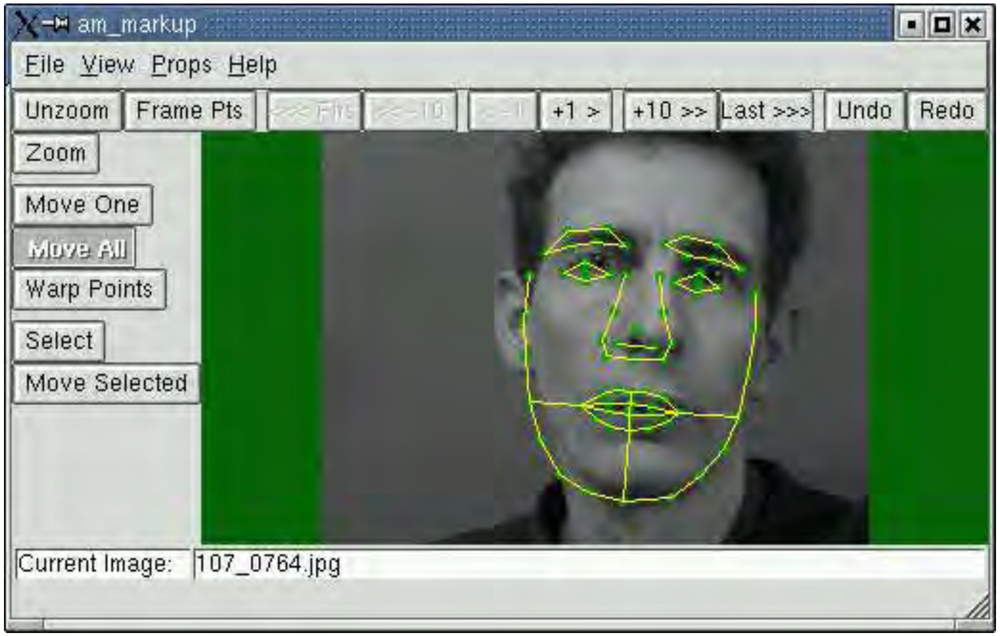


http://www.deformetrica.org/

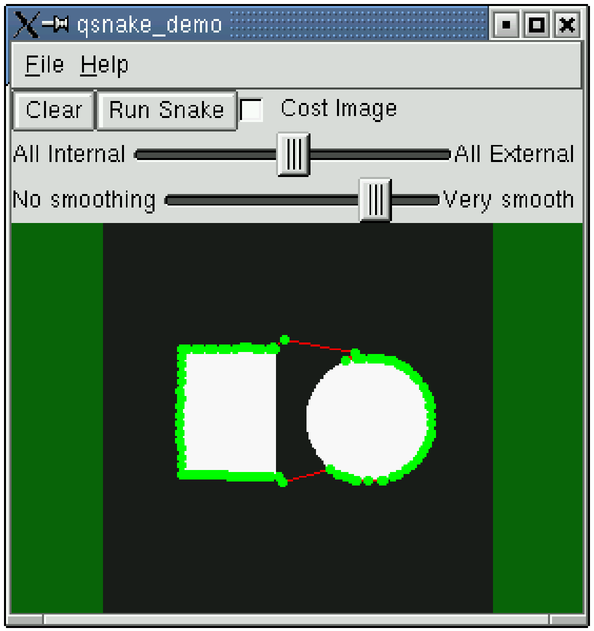
Paper: Durrleman et al., Neuroimage 2014

Software Resources

Tim Cootes, Modeling and Search Software (C++ and VXL)



A set of tools to build and play with Appearance Models and AAMs.



A basic program to experiment with Active Contour Models (snakes).

http://www.isbe.man.ac.uk/~bim/software/index.html

Conclusions

- “Shape” is a fundamental concept of human perception.
- “Shape Analysis” is still a very active research topic.
- “Shape” is an essential concept for Medical Image Analysis.
- Many methods (SSMs in medicine, face & fingerprint recognition, face indexing,…) have found applications in daily routine.
- Serious mathematical & statistical concepts help to make the field much more mature, but:
- Need to bridge the gap between the “Beauty of Math” and “Biological Shape”.

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- **NIH NIBIB 1R01EB014346-01**: ITK-SNAP
- **NIH NINDS R01 HD067731-01A1**: Down's Syndrome
- **NIH P01 DA022446-011:** Neurobiological Consequences of Cocaine Use

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- **and many others ….. (see citations)**

Are you still in "Good Shape"?

Bob Dylan & the Band:
The Shape I'm In

Go out yonder, peace in the valley
Come downtown, have to rumble in the alley

Oh, you don't know the shape I'm in

Has anybody seen my lady
This living alone will drive me crazy

Oh, you don't know the shape I'm in

I'm gonna go down by the wa - ter
But I ain't gonna jump in, no, no
I'll just be looking for my mak - er
And I hear that that's where she's been? oh
Out of nine lives, I spent seven
Now, how in the world do you get to heaven

Oh, you don't know the shape I'm in